Linear theory of ionization cooling in 6D phase space

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A linear theory is developed for ionization cooling of muon beams in periodic channels that can provide cooling of the transverse emittances and also of the longitudinal emittance via the emittance exchange. The channels incorporate solenoids and quadrupoles for transverse focusing, dipoles to generate dispersion, wedged absorbers for ionization, and rf cavities for acceleration. The beam evolution near equilibrium is described by coupled first-order differential equations for five generalized emittances with two excitation sources. The results should be useful for understanding cooling process and for designing cooling channels of future muon colliders and neutrino factories.

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The physics potentials of neutrino factories and muon colliders have stimulated worldwide studies of the feasibility of high-energy muon accelerators [1, 2]. The biggest challenge is to reduce the 6D phase-space volume of a muon beam by orders of magnitude within a fraction of the muons' decay time. Ionization cooling has been proposed as the most promising candidate for this purpose [3, 4]. The principle of ionization cooling is similar to that of synchrotron radiation damping in electron storage rings [5] and arises from the fact that a particle's momentum loss is parallel to the momentum while the acceleration is in the forward direction. Ionization cooling in solenoidal focusing channels has been shown to be effective in reducing the transverse emittances of muon beams [1, 2]. However, ionization cooling of the longitudinal emittance is not straightforward because the derivative of the energy loss with respect to the muon momentum is either too small or of negative sign. Longitudinal cooling may be achieved in an *emittance-exchange* scheme [6] in which dispersions are introduced to transversely separate muons of different momenta, and then wedged (thickness varying transversely) absorbers are used to reduce momentum spread. Studies of ionization cooling in full 6D phase space have mainly relied on simulations due to the complexity of the problem. In this Letter, we develop an analytic, linear theory of beam evolution in 6D phase space during ionization cooling in periodic channels. We use moment-equation approach which is well-established in studying beam dynamics. Some previous applications to ionization cooling are in the Refs. [7–11].

Assuming that the dissipative force from material interaction is weak, the beam moments near equilibrium can be described in terms of the envelope functions determined by the Hamiltonian forces (from the magnets and rf) and a set of generalized emittances [12, 13]. A linear theory of transverse ionization cooling in 4D phase space in periodic, axially symmetric solenoidal channels was developed previously as an evolution of two generalized emittances—the transverse emittance and the angular momentum [9, 10]. In the 6D case, there are five generalized emittances—the two transverse emittances, the longitudinal emittance, the angular momentum, and an x-y correlation. We derive a set of coupled first-order differential equations for the generalized emittances that describe the effects of damping, emittance exchange, and heating due to multiple scattering and energy straggling. The equations turn out to be simple and should be useful.

To begin, we introduce the phase-space vector $X = (x, p_x, y, p_y, z, \delta)^T$, where x, p_x, y and p_y are the muon's transverse coordinates and canonical momenta relative to the reference particle with momentum p_0; and z and $\delta = (p - p_0)/p_0$ are the longitudinal coordinate and momentum deviation, respectively. The equation of motion using path-length s as the time variable is of the form

$$\frac{dX}{ds} = JHX + \left.\frac{dX}{ds}\right|_{\mathrm{M}}. \quad (1)$$

Here, the first term on the right-hand side is the Hamiltonian part of the motion, where J is the simplectic matrix whose elements are the Poisson brackets of the phase-space variables, and H is the symmetric matrix associated with the Hamiltonian H via $\mathsf{H} = X^T HX/2$. The last term in Eq. (1) represents the interaction with materials giving rise to weak dissipation and diffusion.

The Hamiltonian considered in this Letter is

$$\mathsf{H} = \underbrace{\frac{1}{2}\left(p_x^2 + p_y^2\right)}_{\text{drift}} + \underbrace{\frac{1}{2}\kappa(s)^2\left(x^2 + y^2\right) - \kappa(s)\left(xp_y - yp_x\right)}_{\text{solenoid}} \underbrace{- \frac{x\delta}{\rho(s)} + \frac{x^2}{2\rho(s)^2}}_{\text{dipole}} + \underbrace{\frac{1}{2}\,g(s)\left(x^2 - y^2\right)}_{\text{quadrupole}} + \underbrace{\frac{1}{2}\left[\frac{1}{\gamma_0^2}\delta^2 + V(s)z^2\right]}_{\text{longitudinal and rf}}. \quad (2)$$

Here $\kappa(s) = \frac{q}{2p_0}B_s(0,0,s)$ is the normalized on-axis solenoid field strength, where q is muon's charge; $\rho(s) = \frac{q}{p_0}B_y(0,0,s)$ is the radius of curvature of the reference trajectory; $g(s) = \frac{q}{p_0}\frac{\partial B_y}{\partial x}$ is the quadrupole gradient; γ_0 is the Lorentz factor of the reference particle; and $V(s)$ represents longitudinal focusing from rf. The quadrupole gradient is chosen as $g(s) = -1/2\rho(s)^2$ so that the net focusing due to the solenoids, dipoles, and quadrupoles in the x- and y-directions have the same strength $K(s) = \kappa(s)^2 + 1/2\rho(s)^2$. Symmetric focusing is preferred since the main solenoid field continuously rotates the beam and tends to make it symmetric.

As shown for transverse cooling [10], beam motion in the Larmor frame is much simpler because transverse coupling from the angular momentum term is removed. Rotating to the Larmor frame, the Hamiltonian becomes

$$\tilde{\mathsf{H}} = \frac{1}{2}(\tilde{p}_x^2 + \tilde{p}_y^2) + \frac{K}{2}(\tilde{x}^2 + \tilde{y}^2) - \frac{(\tilde{x}\cos\phi + \tilde{y}\sin\phi)\delta}{\rho(s)} + \mathrm{rf}. \tag{3}$$

Here the symbol ˜ indicates quantities in the Larmor frame, which is rotating with the angle $\phi(s) = \int_0^s \kappa(\bar{s})d\bar{s}$. To further simplify, we decouple the transverse and longitudinal motions by introducing the dispersion functions $\tilde{D}_x$ and $\tilde{D}_y$, and the canonical transformation [14]

$$\tilde{x} = \tilde{x}_\beta + \tilde{D}_x\delta \ , \quad \tilde{p}_x = \tilde{p}_{x_\beta} + \tilde{D}'_x\delta, \quad (x \leftrightarrow y) \tag{4}$$

$$z = \hat{z} - \tilde{D}'_x\tilde{x} + \tilde{D}_x\tilde{p}_x - \tilde{D}'_y\tilde{y} + \tilde{D}_y\tilde{p}_y \ , \quad \delta = \hat{\delta}. \tag{5}$$

Hereafter a prime indicates differentiation with respect to s. By requiring the dispersions to satisfy the equations

$$\tilde{D}''_x + K\tilde{D}_x = \frac{\cos\phi}{\rho} \ , \quad \tilde{D}''_y + K\tilde{D}_y = \frac{\sin\phi}{\rho}, \tag{6}$$

and to be zero at the rf cavities, the transverse and longitudinal motions are decoupled with the new Hamiltonian

$$\tilde{\mathsf{H}}_\beta = \frac{1}{2}\left(\tilde{p}_{x_\beta}^2 + \tilde{p}_{y_\beta}^2\right) + \frac{1}{2}K(\tilde{x}_\beta^2 + \tilde{y}_\beta^2) + \frac{1}{2}(I\delta^2 + \mathrm{V}\hat{z}^2)\ , \tag{7}$$

where $I(s) = \frac{1}{\gamma_0^2} - \frac{\tilde{D}_x\cos[\phi(s)]}{\rho(s)} - \frac{\tilde{D}_y\sin[\phi(s)]}{\rho(s)}$.

The material part of Eq. (1) is of the form

$$\left.\frac{dX}{ds}\right|_{\mathrm{M}} = \left.\frac{dX}{ds}\right|_{\mathrm{M,D}} + \Xi = AX + \Xi. \tag{8}$$

Here $(dX/ds)|_{\mathrm{M,D}}$ is the dissipative part of the interaction with material, A is the dissipation matrix, and Ξ represents the stochastic excitations discussed later. The dissipative part of the equation of motion is given by

$$\left.\frac{dx}{ds}\right|_{\mathrm{M,D}} = \left.\frac{dy}{ds}\right|_{\mathrm{M,D}} = \left.\frac{dz}{ds}\right|_{\mathrm{M,D}} = 0, \tag{9}$$

$$\left.\frac{dp_x}{ds}\right|_{\mathrm{M,D}} = -\eta\,(p_x + \kappa y)\,, \tag{10}$$

$$\left.\frac{dp_y}{ds}\right|_{\mathrm{M,D}} = -\eta\,(p_y - \kappa x)\,, \tag{11}$$

$$\left.\frac{d\delta}{ds}\right|_{\mathrm{M,D}} = -(\partial_\delta\eta)\delta - (\partial_x\eta)x - (\partial_y\eta)y. \tag{12}$$

Here $\eta = \frac{1}{pv}\frac{dE}{ds}$ is a positive quantity characterizing the average force due to ionization energy loss for a muon of momentum p and velocity v. The terms $(p_x + \kappa y)$ and $(p_y - \kappa x)$ are, respectively, the x- and y-components of the kinetic momentum. The wedged absorbers are treated as having uniform thickness with density depending linearly on the transverse coordinates. To linear order, the energy dependence of ionization energy loss is given by $\partial_\delta\eta$. The simple model in Eqs. (10, 11) has been shown to work well for transverse cooling [10].

The matrix A in Eq. (8) can be decomposed into two parts $A = A_H + A_D$, where $A_H = (A + JA^TJ)/2$. The matrix A_H can be considered as belonging to the Hamiltonian part since it is of the form J times a symmetric matrix [13]. We drop this part and use only A_D as the dissipation matrix. (We may assume that the Hamiltonian contains negligibly small additional terms that cancel the terms due to A_H.) Then Eqs. (9-12) becomes

$$\left.\frac{dx}{ds}\right|_{\mathrm{M,D}} = -\frac{1}{2}\eta x, \tag{13}$$

$$\left.\frac{dp_x}{ds}\right|_{\mathrm{M,D}} = -\frac{1}{2}\eta p_x - \eta\kappa y + \frac{1}{2}(\partial_x\eta)z, \tag{14}$$

$$\left.\frac{dz}{ds}\right|_{\mathrm{M,D}} = -\frac{1}{2}(\partial_\delta\eta)z, \tag{15}$$

$$\left.\frac{d\delta}{ds}\right|_{\mathrm{M,D}} = -\frac{1}{2}\left[(\partial_\delta\eta)\delta + (\partial_x\eta)x + (\partial_y\eta)y\right]. \tag{16}$$

The equations for the y plane are the same as those for the x except a positive sign for the $\eta\kappa x$ term.

The phase-space distributions relevant in linear approximation are Gaussian distributions that can be specified by the quadratic beam-moment matrix $\Sigma = \langle XX^T\rangle$, where the brackets indicate the averaging. From the equation of motion we have the moment equation

$$\frac{d\Sigma}{ds} = (JH + A_D)\Sigma + \Sigma\,(JH + A_D)^T + B. \tag{17}$$

Here the diagonal matrix $B = \mathrm{diag}(0, \chi, 0, \chi, 0, \chi_\delta)$ arises from the stochastic excitations represented by Ξ in Eq. (8). There are two different sources of excitations: multiple scattering characterized by the projected mean-square angular deviation per unit length

$\chi = \left(\frac{13.6\,\mathrm{MeV}}{pv}\right)^2 \frac{1}{L_{\mathrm{rad}}}$, where L_{rad} is the radiation length of absorbers, and energy straggling characterized by the mean-square relative energy deviation per unit length χ_δ.

We need to make two changes of variables as in the case of the Hamiltonian part of the motion. The change to the Larmor frame is easy; since the muons' local interaction with material is isotropic, Eqs. (13-17) will not be changed by a rotation and thus apply to the variables $(\tilde{x}, \tilde{p}_x, \tilde{y}, \tilde{p}_y, z, \delta)$. Changing variables to $(\tilde{x}_\beta, \tilde{p}_{x_\beta}, \tilde{y}_\beta, \tilde{p}_{y_\beta}, \hat{z}, \delta)$ via Eqs. (4, 5) is straightforward but the resulting equations are cumbersome and not written down explicitly. From now on, we will drop the $\tilde{}$ symbol to simplify the notation.

The moment equation is formidable since it represents a coupled evolution of the 21 independent moments in the symmetric 6×6 matrix Σ. However, the system becomes greatly simplified if consideration is limited to the behavior near equilibrium, as we do in the rest of this Letter. It is physically reasonable to assume that the moment matrix at equilibrium is a periodic function of s with the periodicity of the cooling channel. If the dissipative forces are weak, the distribution function, which is of Gaussian shape for the linear system under consideration, must also evolve approximately as in the Hamiltonian system. The distribution function can thus be specified by a set of quadratic single-particle invariants with periodic coefficients. In the present case, from the decoupled Hamiltonian H_β, we find the following five linearly-independent quadratic invariants:

$$I_x = \gamma_T\, x_\beta^2 + 2\alpha_T\, x_\beta p_{x_\beta} + \beta_T\, p_{x_\beta}^2, \tag{18}$$

$$I_y = \gamma_T\, y_\beta^2 + 2\alpha_T\, y_\beta p_{y_\beta} + \beta_T\, p_{y_\beta}^2, \tag{19}$$

$$I_z = \gamma_L\, \hat{z}^2 + 2\alpha_L\, \hat{z}\delta + \beta_L\, \delta^2, \tag{20}$$

$$I_{xy} = \gamma_T\, x_\beta y_\beta + 2\alpha_T \frac{x_\beta p_{y_\beta} + y_\beta p_{x_\beta}}{2} + \beta_T\, p_{x_\beta} p_{y_\beta}, \tag{21}$$

$$L_z = x_\beta p_{y_\beta} - y_\beta p_{x_\beta}. \tag{22}$$

Here the envelope functions, γ_T, etc., are the periodic solution of the following equations

$$\beta_T' = -2\alpha_T\,,\ \alpha_T' = K\beta_T - \gamma_T\,,\ \gamma_T = \frac{1+\alpha_T^2}{\beta_T} \tag{23}$$

and

$$\beta_L' = -2I\alpha_T\,,\ \alpha_L' = V\beta_T - I\gamma_T\,,\ \gamma_L = \frac{1+\alpha_L^2}{\beta_L}. \tag{24}$$

In the above, I_x, I_y, and I_z are the familiar Courant-Snyder (C-S) type invariants [15] for each of the three degrees of freedom; L_z is the angular momentum; and I_{xy} is the invariant obtained by taking Poisson bracket of L_z and I_x. Note that I_x, I_y, and I_{xy} are associated with the same set of the C-S parameters $\gamma_T, \alpha_T, \beta_T$ reflecting the degeneracy of the x-y part of the Hamiltonian H_β. The four transverse invariants were discussed in the context of an isotropic harmonic oscillator [16]. The five invariants form a complete set of the quadratic invariants.

Averaged over the phase space, these five single-particle invariants lead to five beam invariants that are usually called beam emittances:

$$\epsilon_i = \frac{1}{2}\langle I_i\rangle, \qquad i \in \{x, y, z, xy, L\}. \tag{25}$$

Using emittances and invariants, the normalized equilibrium distribution can be written as

$$\rho(X) = \frac{1}{(2\pi)^3\epsilon_{6\mathrm{D}}}\, e^{-\frac{\epsilon_y I_x + \epsilon_x I_y - 2\epsilon_{xy} I_{xy} - 2\epsilon_L L_z}{2(\epsilon_x\epsilon_y - \epsilon_{xy}^2 - \epsilon_L^2)} - \frac{I_z}{2\epsilon_z}}, \tag{26}$$

where the 6D emittance is

$$\epsilon_{6\mathrm{D}} = (\epsilon_x\epsilon_y - \epsilon_{xy}^2 - \epsilon_L^2)\,\epsilon_z. \tag{27}$$

We can then compute the corresponding nonzero moments as follows:

$$\left(\langle x_\beta^2\rangle, \langle x_\beta p_{x_\beta}\rangle, \langle p_{x_\beta}^2\rangle\right) = \epsilon_x\,(\beta, -\alpha, \gamma)_T\,, \quad (x \leftrightarrow y) \tag{28}$$

$$\left(\langle x_\beta y_\beta\rangle, \frac{\langle x_\beta p_{y_\beta} + y_\beta p_{x_\beta}\rangle}{2}, \langle p_{x_\beta} p_{y_\beta}\rangle\right) = \epsilon_{xy}(\beta, -\alpha, \gamma)_T, \tag{29}$$

$$\left(\langle x_\beta p_{y_\beta}\rangle, \langle y_\beta p_{x_\beta}\rangle\right) = \epsilon_L(1, -1), \tag{30}$$

$$\left(\langle\hat{z}^2\rangle, \langle\hat{z}\delta\rangle, \langle\delta^2\rangle\right) = \epsilon_z\,(\beta_L, -\alpha_L, \gamma_L)\,. \tag{31}$$

These equations may be viewed as the inverse of Eq. (25).

A general study of the equilibrium state for a weakly dissipative, periodic system was carried out previously based on orthogonal expansions in the linear space of the moments [13]. These authors pointed out the existence of five invariants for systems with x-y degeneracy.

We now consider the system near but not at the equilibrium due to interaction with material. The approach of this system to the equilibrium may be described by a slow s-dependence of the generalized emittances. The s-derivatives can be computed by inserting the material part of the equation of motion, Eqs. (13-16), into the derivative of Eq. (25) and rearranging the results with Eqs. (28-31). In doing this we note that the Hamiltonian forces do not contribute and the betatron and synchrotron motions are decoupled. The stochastic contributions can be derived from Eq. (17). The results are

$$\epsilon_s' = -(\eta - ec_-)\epsilon_s + ec_+\epsilon_a + es_+\epsilon_{xy} + b\,\epsilon_L + \chi_s, \tag{32}$$

$$\epsilon_a' = -(\eta - ec_-)\epsilon_a + ec_+\epsilon_s + \chi_a, \tag{33}$$

$$\epsilon_{xy}' = -(\eta - ec_-)\epsilon_{xy} + es_+\epsilon_s + \chi_{xy}, \tag{34}$$

$$\epsilon_L' = -(\eta - ec_-)\epsilon_L + b\,\epsilon_s + \chi_L, \tag{35}$$

$$\epsilon_z' = -(\partial_\delta\eta + 2ec_-)\epsilon_z + \chi_z, \tag{36}$$

where ϵ_s and ϵ_a are the symmetric and asymmetric emittances $(\epsilon_x \pm \epsilon_y)/2$, $e = |\vec{D}|\cdot|\vec{\partial\eta}|/2$ is half of the maximum exchange rate through dispersions and wedges, $c_\pm = \cos(\theta_D \pm \theta_W)$ and $s_\pm = \sin(\theta_D \pm \theta_W)$ with θ_D and θ_W being the orientations of the dispersion vector and the wedges, and $b = \eta\kappa\beta_T + \alpha_T e s_- + \beta_T e' s_-'$ with $e' = |\vec{D}'|\cdot|\vec{\partial\eta}|/2$ and $s_-' = \sin(\theta_{D'} - \theta_W)$. The excitation terms are

$$\chi_s = \frac{1}{2}\beta_T\chi + \frac{1}{2}\mathcal{H}_s\,\chi_\delta, \tag{37}$$

$$\chi_a = \frac{1}{2}\mathcal{H}_a\,\chi_\delta, \tag{38}$$

$$\chi_z = \frac{1}{2}\beta_L\chi_\delta + \frac{1}{2}\gamma_L(D_x^2 + D_y^2)\chi, \tag{39}$$

$$\chi_{xy} = \frac{1}{2}\mathcal{H}_{xy}\,\chi_\delta, \tag{40}$$

$$\chi_L = \frac{1}{2}\mathcal{H}_L\chi_\delta. \tag{41}$$

Here the $\mathcal{H}$ functions are defined similarly as Eqs. (18-22) but replacing phase-space variables with dispersion functions. For example, as in radiation damping theory, $\mathcal{H}_x = \gamma_T D_x^2 + 2\alpha_T D_x D_x' + \beta_T D_x'^2$. These heating terms arise from stochastic contribution to the beam invariants. For instance, multiple scattering and energy straggling cause the momenta to fluctuate with $\langle p_x^2\rangle = \langle p_y^2\rangle = \chi$ and $\langle\delta^2\rangle = \chi_\delta$, but there are no correlated position fluctuations. Thus they contribute to the invariants through only the $\beta\langle p^2\rangle$ term and yield the $\beta_T\chi$ and $\beta_L\chi_\delta$ terms. Meanwhile, straggling causes correlated fluctuations in transverse position and momentum via Eq. (4), thus yields the four $\mathcal{H}\,\chi_\delta$ terms that have the invariant structures.

Note that the emittance exchange is accomplished by trading the damping rate ec_- between the transverse and longitudinal degrees of freedom. Without excitations,

$$\frac{d\,\epsilon_{6\mathrm{D}}}{ds} = -\,(2\eta + \partial_\delta\eta)\,\epsilon_{6\mathrm{D}}. \tag{42}$$

Therefore the total 6D damping rate is independent of the emittance exchange. This is equivalent to the Robinson theorem for radiation damping [5].

Let us make a few observations on the emittance evolution equations, Eqs. (32-36). First, as the dispersions go to zero, they reduce to our previous result on the transverse cooling in straight solenoid channels [10]. Second, the longitudinal and transverse evolutions are decoupled (except exchanging the damping rate). Hence the longitudinal evolution can be analytically integrated. Third, because the emittances will not change much in one period, it should be a good approximation to average the evolution equation over one period. After averaging, the emittances can be solved by straightforward diagonalization. Particularly, the equilibrium longitudinal and symmetric transverse emittances are then given by

$$\epsilon_z^{\mathrm{eq.}} \simeq \overline{\chi}_z \,/\, \overline{\partial_\delta\eta + 2ec_-}\,, \tag{43}$$

$$\epsilon_s^{\mathrm{eq.}} \simeq \frac{\overline{\eta - ec_-}\;\overline{\chi}_s + \overline{ec_+}\;\overline{\chi}_a + \overline{es_+}\;\overline{\chi}_{xy} + \overline{b}\;\overline{\chi}_L}{\overline{\eta - ec_-}^{\,2} - \overline{ec_+}^{\,2} - \overline{es_+}^{\,2} - \overline{b}^{\,2}}. \tag{44}$$

Here the overline indicates averaging over a period. Fourth, to achieve the maximum longitudinal cooling, the ec_- term needs to be maximized by increasing dispersion, the number of wedges, and wedge angle, and by orienting the wedge along the dispersion vector (i.e., $\Delta\theta = \theta_D - \theta_W = 0$). If the wedges are placed at dispersion maxima with $\Delta\theta = 0$, the term b reduces to $b = \eta\kappa\beta_T$, which can be designed to average to zero. Fifth, it is easy to see, from the excitation terms, that an obvious way to limit heating is to reduce the transverse and longitudinal beta functions at the absorbers. It is also important to optimize the dispersions to balance the needs of large emittance exchange and small excitations.

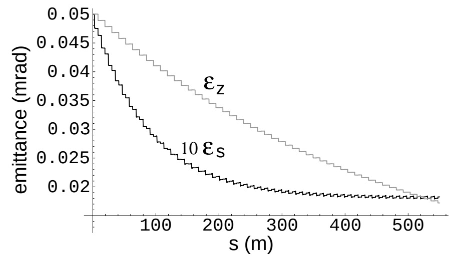

FIG. 1: Transverse and longitudinal emittance evolution

As an example, we consider a 6D cooling channel modified from the first section of the "SFOFO" transverse cooling channel used in the feasibility study-II [2]. About 20cm dispersion is introduced in the middle of the 5.5m solenoid cooling cells. Lithium-hydride wedged absorbers with 90° vertex are placed at the dispersion maximum to obtain emittance exchange. Using Eqs. (32-36), we tracked the emittances over 500m for an axisymmetric incoming beam with $\epsilon_s = 5$ mm·rad and $\epsilon_z = 50$ mm·rad, which is close to the feasibility study values. Figure 1 shows the evolution of ϵ_s and ϵ_z. The other three emittances are orders of magnitude smaller. The transverse emittance has damped to its equilibrium value while the longitudinal emittance is still far from its equilibrium value of 1.8 mm·rad. This is partly due to the large initial longitudinal emittance and partly because the longitudinal damping coefficient is only 20% of the transverse one.

In closing, we developed a linear theory of 6D ionization cooling that should be useful for understanding the cooling process and for initial evaluation of cooling channels. The next step is to include the nonlinearity and develop practical designs, a subject of ongoing research.

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