# MUCOOL-243

# Transverse Emittance Increase by Energy-loss Straggling and Dispersion

**David Neuffer**
**Fermilab**

**Abstract**

We note that random energy loss at non-zero dispersion can increase the transverse emittance, We derive expressions for this effect in ionization cooling, and compare it with multiple scattering emittance heating. The effect can be large when cooling with absorbers located at large dispersion and small $\beta_\perp$, and when cooling is attempted at larger energies.

## Introduction

In this note we discuss the beam heating processes involved in ionization cooling. In ionization cooling ($\mu$-cooling), particles pass through a material medium and lose energy (momentum) through ionization interactions, and this is followed by beam reacceleration in rf cavities.[1, 2, 3, 4, 5, 6] The losses are parallel to the particle motion, and therefore include transverse and longitudinal momentum losses; the reacceleration restores only longitudinal momentum. The loss of transverse momentum reduces particle emittances, cooling the beam. However, the random process of multiple scattering in the material medium increases the rms beam divergence, adding a heating term.

The basic differential equation for rms transverse cooling is:

$$\frac{d\varepsilon_N}{ds} = -\frac{1}{\beta^2 E}\frac{dE}{ds}\varepsilon_N + \frac{\beta\gamma\ \beta_\perp}{2}\frac{d\langle\theta_{rms}{}^2\rangle}{ds} = -\frac{1}{\beta^2 E}\frac{dE}{ds}\varepsilon_N + \frac{\beta_\perp E_s^2}{2\beta^3 m_\mu c^2 L_R E} \tag{1}$$

where the first term is the energy-loss cooling effect and the second is the multiple scattering heating term. Here $\varepsilon_N$ is the normalized emittance, E is the beam energy, $\beta$ = v/c and $\gamma$ are the usual kinematic factors, dE/ds is the energy loss rate, $\theta_{rms}$ is the rms multiple scattering angle, $L_R$ is the material radiation length, $\beta_\perp$ is the betatron function, and $E_s$ is the characteristic scattering energy (~13.6 MeV).[6] (The normalized emittance is related to the geometric emittance $\varepsilon_\perp$ by $\varepsilon_N = \varepsilon_\perp(\beta\gamma)$, and the beam size is given by $\sigma_\perp = (\varepsilon_\perp\beta_\perp)^{1/2}$.)

Ionization cooling does not directly provide longitudinal cooling. The equation for longitudinal cooling with energy loss is:

$$\frac{d\sigma_E^2}{ds} = -2\frac{\partial\frac{dE}{ds}}{\partial E}\sigma_E^2 + \frac{d\langle\Delta E_{rms}{}^2\rangle}{ds} \tag{2}$$

in which the first term is the cooling term and the second is the heating term caused by random fluctuations in the particle energy (energy straggling). This heating term is given approximately by:

$$\frac{d\langle\Delta E_{rms}{}^2\rangle}{ds} = 4\pi\left(r_e m_e c^2\right)^2 n_e\gamma^2\left(1-\frac{\beta^2}{2}\right) \cong 0.157\rho\frac{Z}{A}\gamma^2\left(1-\frac{\beta^2}{2}\right)\ \ (\mathrm{MeV})^2\,\mathrm{cm}^2/\mathrm{gm}, \tag{3}$$

where $n_e$ is the electron density in the material ($n_e=N_A\rho Z/A$).

Longitudinal beam cooling can occur if the derivative $\partial(dE/ds)/\partial E > 0$. This energy loss can be estimated by the Bethe-Bloch equation[7]:

$$\frac{dE}{ds} = 4\pi N_A r_e^2 m_e c^2 \rho \ \frac{Z}{A}\left[\frac{1}{\beta^2}\ln\left(\frac{2m_e c^2\gamma^2\beta^2}{I}\right) - 1 - \frac{\delta}{2\beta^2}\right] \tag{4}$$

where $N_A$ is Avogadro's number, $\rho$, A and Z are the density, atomic weight and number of the absorbing material, $m_e$ and $r_e$ are the mass and classical radius of the electron, ($4\pi N_A r_e^2 m_e c^2$ = 0.3071 MeV cm$^2$/gm). The ionization constant I is approximately 16 $Z^{0.9}$ eV, and $\delta$ is the density effect factor, which is small for low-energy $\mu'$s. The derivative of this energy loss is negative (or naturally heating) for $E_\mu < \sim 0.3$ GeV, and is only slightly positive for higher energies.

However, the derivative can be enhanced by placing the absorbers where transverse position depends upon energy (nonzero dispersion) and where the absorber density or thickness also depends upon energy, such as in a wedge absorber. In that case the cooling derivative is rewritten as:

$$\frac{\partial \frac{dE}{ds}}{\partial E} \Rightarrow \left.\frac{\partial \frac{dE}{ds}}{\partial E}\right|_0 + \frac{dE}{ds}\frac{\eta\rho'}{\beta c p \rho_0} = g_L \frac{\frac{dp}{ds}}{p} \tag{5}$$

where $\rho'/\rho_0$ is the relative change in density with respect to transverse position, $\rho_0$ is the reference density associated with dE/ds, and $\eta$ is the dispersion ($\eta$ = d x /d($\Delta$p/p)). The partition number $g_L$ describes the cooling rate related to the mean momentum loss, and the wedge configuration increases the longitudinal partition number by $\eta\rho'/\rho_0$. It also decreases the corresponding transverse partition number by the same amount: $g_x \to (1-\eta\rho'/\rho_0)$, which decreases the transverse cooling. The sum of the cooling rates or partition numbers (over x, y, and L) remains constant; a similar invariant sum of cooling rates, with emittance exchange from radiation at nonzero dispersion, occurs in radiation damping of electrons.

In radiation damping, fluctuations in energy loss at nonzero dispersion perturb the transverse motion, and thereby increase the transverse emittance. This effect is the dominant transverse heating effect in radiation cooling. The same type of heating effect also occurs in ionization cooling, when random energy loss occurs at non-zero dispersion. In previous studies this particular heating term was not explicitly included, since the bulk of the energy-loss cooling was expected to occur at zero dispersion, and the multiple-scattering heating term (see eq. 1) is much larger, and (as discussed below) it remains much larger with energy loss at moderate dispersions. More recent studies by C.-X. Wang and K-J. Kim have included expressions for this and other coupling effects in their more complete treatments of 6-D ionization cooling theory.[8, 9].

However, recent studies of ionization cooling have included extended insertions with energy loss at non-zero dispersion, and dispersive random energy loss can cause nontrivial transverse emittance increase. In this note we calculate the size of this transverse heating term, adding it to the cooling formalism, and compare it with the multiple scattering heating terms. Design guidelines to limit the heating effects are discussed.

### Calculation of the heating effect

The present discussion follows similar treatments on the similar effect in radiation damping. We consider the effects of a random energy loss $\delta E$, leading to a change of momentum $\delta_p = \delta p/p =$

$1/\beta^2\ \delta E/E$. We assume that the random energy fluctuation occurs at a point of the beam transport lattice with dispersion $\eta$, $\eta'$, and betatron functions $\beta_x$, $\alpha_x$, and $\gamma_x = (1+\alpha_x^2)/\beta_x$. We express the transverse motion in terms of the invariant amplitude function $a_x$ where:

$$a_x^2 = \gamma_x x^2 + \alpha_x x x' + \beta_x x'^2 \tag{6}$$

x and $x'$ =dx/ds are the local transverse motion coordinates, and we have placed the dispersion in the x-plane. When a particle changes $\delta p/p$ by a random amount $\delta$, its position and transverse velocity do not change. However the dispersive part of its position changes by $\eta\delta$ and its transverse velocity changes by $\eta'\delta$, and that implies the transverse motion parameters x, $x'$ change by $-\eta\delta$ and $-\eta'\delta$, respectively. Since the random energy loss is uncorrelated with position the rms change in amplitude can be written as:

$$\Delta\left\langle a_x^2 \right\rangle = \Delta\left\langle \gamma_x x^2 + \alpha_x x x' + \beta_x x'^2 \right\rangle = \left(\gamma_x \eta^2 + \alpha_x \eta \eta' + \beta_x \eta'^2\right)\delta^2 = H\delta^2 \tag{7}$$

where we have introduced the parameter H = $\gamma_x \eta^2 + \alpha_x \eta \eta' + \beta_x \eta'^2$, that was previously used in radiation cooling.

From the relationship between amplitude and rms normalized emittance, that is given by: $\varepsilon_{N,rms} = \beta\gamma \langle a^2\rangle/2$, we can find an expression for the increase in emittance:

$$\frac{d\varepsilon_N}{ds} = \frac{\left\langle \gamma_x \eta^2 + \alpha_x \eta \eta' + \beta_x \eta'^2 \right\rangle}{2} \frac{\beta\gamma}{\beta^4 E_\mu{}^2} \frac{d\left\langle \Delta E_{rms}{}^2 \right\rangle}{ds} \tag{8}$$

Including eq. 3 we can rewrite this as:

$$\frac{d\varepsilon_N}{ds} = \frac{\left\langle \gamma_x \eta^2 + \alpha_x \eta \eta' + \beta_x \eta'^2 \right\rangle}{2} \frac{\beta\gamma}{\beta^4 E_\mu{}^2} 4\pi\left(r_e m_e c^2\right)^2 n_e \gamma^2 \left(1 - \frac{\beta^2}{2}\right) \tag{9}$$

It is useful to compare this term for transverse emittance increase through straggling at non-zero dispersion with the multiple-scattering induced emittance heating. This comparison is somewhat material-dependent through $L_R$, $\rho$, Z, A. For low-Z materials used in ionization cooling we use the following properties:

For $H_2$, $\rho$ = 0.071 gm/cm$^3$, Z=1, A=1, $L_R$ =865cm, dE/ds = 0.292 MeV/cm,
For LiH, $\rho$ = 0.79 gm/cm$^3$, Z=1+3, A=1+7, $L_R$ =102cm, dE/ds = 1.34 MeV/cm,
For Be, $\rho$ = 1.85 gm/cm$^3$, Z=4, A=9, $L_R$ =35.3cm, dE/ds = 2.98 MeV/cm.

To simplify the comparisons we set the beam at a waist in dispersion and betatron motion ($\eta'$=0, $\alpha_x$ = 0, $\gamma_x=1/\beta_x$). (This should be the optimal location for cooling absorbers.) We also use the relativistic approximation $\beta$=1. Under these assumptions, the heating terms can be evaluated:

For $H_2$, the multiple scattering heating term is $d\varepsilon_N/ds \cong 9.6\times10^{-6}\ \beta_x/\gamma$, and the straggling-based heating term is $d\varepsilon_N/ds \cong 2.5\times10^{-7}\ \gamma\eta^2/\beta_x$.

For LiH, the multiple scattering heating term is $d\varepsilon_N/ds \cong 8.1\times10^{-5}\ \beta_x/\gamma$, and the straggling-based heating term is $d\varepsilon_N/ds \cong 1.4\times10^{-6}\ \gamma\eta^2/\beta_x$.

For Be, the multiple scattering heating term is $d\varepsilon_N/ds \cong 2.35\times10^{-4}\ \beta_x/\gamma$, and the straggling-based heating term is $d\varepsilon_N/ds \cong 2.9\times10^{-6}\ \gamma\eta^2/\beta_x$.

In each of these examples the scattering-based heating term is larger, unless $\eta^2 >> \beta_x^2/\gamma^2$ The crossover occurs at $\eta \cong 6\beta_x/\gamma$ for hydrogen absorbers, and at $\eta \cong 9\beta_x/\gamma$ for Be. Note that this is a weighted average over all absorbers; a cooling system that has most of its absorbers at zero-dispersion inserts would be scattering dominated.

## Examples

In recent studies, emittance exchange sections have been developed which include absorbers at regions of nonzero dispersion, where dispersion-straggling emittance increase can occur. In this section we explore some suggested examples and indicate the relative importance of this effect.

Kirk, Garren and Fukui have studied a ring cooler in which hydrogen absorbers are placed in low-$\beta_x$ insertions with dispersion, and the absorber ends are wedge-shaped to obtain emittance exchange.[10] In a particular example $\eta$= 45cm, $\eta'$= 0, $\beta_x = \beta_y$ =25cm, and $\alpha_x = \alpha_y = 0$ at the absorber centers. At these parameters, and at $\gamma \cong 2$, the scattering term is ~4× larger than the dispersion term. The dispersion + straggling heating is not negligible, and becomes larger than the scattering term for $\gamma > \sim 4$, or $P_\mu > 400$ MeV/c (with the same $\beta_x$, $\eta$). Relatively small lattice changes could enlarge it to unacceptable levels.

Berg, Fernow, Palmer have an ionization cooling ring with hydrogen absorbers at focus points (waists) where the dispersion $\eta$ is ~8cm, while the corresponding $\beta_\perp$ is ~ 40cm.[11] The scattering heating term is ~250× larger than the dispersion term. The dispersion + straggling heating is almost negligible.

In general the dispersion-straggling effect becomes large with absorbers in regions with large dispersion and small $\beta_\perp$, and also at larger energies (large $\gamma$), where the energy straggling becomes large. The constraint $\eta << 10\beta_\perp/\gamma$ (within the absorbers) should be obeyed to avoid the effect. However, the effect must be included and carefully considered if that constraint is violated, probably using a more complete 6-D ionization cooling formalism, such as that developed by Wang and Kim.[8, 9]

## References

[1] A. N. Skrinsky and V.V. Parkhomchuk, Sov. J. Nucl. Physics **12**, 3(1981).
[2] E. A. Perevedentsev and A. N. Skrinsky, Proc. 12th Int. Conf. on High Energy Accel., 485 (1983).
[3] D. Neuffer, Particle Accelerators **14**, 75 (1983).
[4] D. Neuffer, Proc. 12th Int. Conf. on High Energy Accelerators, 481 (1983).
[5] D. Neuffer, Nucl. Inst. and Meth. **A350**, 27 (1994).
[6] D. Neuffer, CERN 99-14 (1999).
[7] Particle Data Group – C. Caso et al., "Review of Particle Physics", European Physical Journal **C3** (1998).
[8] C.-X. Wang and K.-J. Kim, "Linear theory of Ionization Cooling in 6D Phase Space", MuCool Note 240, April, 2002.
[9] C. X. Wang and K. J. Kim. "Linear theory of Transverse and Longitudinal Cooling in a Quadrupole Channel", MuCOOL 222, October 2001.
[10] A. Garren, H. Kirk, and Y. Fukui, unpublished, 2001.
[11] J. S. Berg, R. C. Fernow and R. B. Palmer, "An Alternating Solenoid Focused Ionization Cooling Ring", MuCool Note 239, March 2002.