VARIATIONS ON A THEME: TOROIDAL LITHIUM LENS

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Abstract

We numerically compute the magnetic field generated by an idealized bent lithium lens using the Biot-Savart Law. The current density vector is of the form $\mathbf{J} = J_o \left\{ \frac{R_o}{R} \right\} \mathbf{e}_\phi$.

1 INTRODUCTION

A lithium lens with an specified curvature has being proposed to be used in cooling rings to achieve transverse and longitudinal emittances appropriate for a Muon Collider [1], [2] and [3]. Recently there have been two papers in the subject: a numerical computation of the field by Fernow [4] and an analytical study by the author [5].

The Biot-Savart expression for the magnetic field is

$$\boxed{\mathbf{B}(\mathbf{r}) = \frac{\mu_o}{4\pi} \int d\,\mathbf{r}' \frac{\mathbf{J} \times (\mathbf{r} - \mathbf{r}')}{|r - r'|^3}} \tag{1}$$

where the integral is over the volume where $\mathbf{J}(\mathbf{r})$ is non-vanishing.

Notice now the vector identity

$$\nabla \times \left(\frac{\mathbf{J}}{|r - r'|} \right) = \frac{1}{|r - r'|} \nabla \times \mathbf{J} + \nabla \left(\frac{1}{|r - r'|} \right) \times \mathbf{J} \tag{2}$$

Now recall that $\nabla \left(\frac{1}{|r-r'|} \right) = -\frac{(\mathbf{r}-\mathbf{r}')}{|r-r'|^3}$; substituting in the identity above and using the fact that $\nabla \times \mathbf{J} = 0$, we can write the Biot-Savart law as

$$\mathbf{B}(\mathbf{r}) = \frac{\mu_o}{4\pi} \int d\,\mathbf{r}' \nabla \times \left(\frac{\mathbf{J}(\mathbf{r}')}{|r - r'|} \right) = \frac{\mu_o}{4\pi} \int d\,S' \frac{\mathbf{e}_r \times \mathbf{J}(\mathbf{r}')}{|r - r'|} \tag{3}$$

where S' is the surface enclosing the volume of the conductor where the current density flows, and $\mathbf{e}_r$ is the unit vector on the surface pointing outward. In the last step we make use of a generalized Gauss' theorem. [6] [7]

In toroidal coordinates (see [5]) the area element is $d\mathbf{S}(r,\theta,\phi) = aRd\,\theta d\,\phi\,\mathbf{e}_r$ with

$$\begin{aligned} R =& R_o + r\cos\theta \\ \mathbf{e}_r =& (\cos\theta'\cos\phi', \cos\theta'\sin\phi', \sin\theta') \\ \mathbf{J} =& (-J_\phi\sin\phi', J_\phi\cos\phi', 0) \end{aligned} \tag{4}$$

The current density is assumed to be of the form $J_\phi = J_o\frac{R_o}{R}$. The Cartesian components of the magnetic field are:

$$\begin{aligned} B_x &= -\frac{\mu_o a R_o J_o}{4\pi}\int d\,\phi' d\,\theta' \frac{\sin\theta'\cos\phi'}{\Omega(r,r')} \\ B_y &= -\frac{\mu_o a R_o J_o}{4\pi}\int d\,\phi' d\,\theta' \frac{\sin\theta'\sin\phi'}{\Omega(r,r')} \\ B_z &= \frac{\mu_o a R_o J_o}{4\pi}\int d\,\phi' d\,\theta' \frac{\cos\theta'}{\Omega(r,r')} \end{aligned} \tag{5}$$

where $\Omega(r,r') = |r - r'|$.

These expressions give the magnetic field for an idealized toroidal lithium lens of length $L = R\Delta\phi$. For completion we give the toroidal components of $\mathbf{B}$, *i.e*

$$\begin{aligned} B_r &= B_x\cos\theta\cos\phi + B_y\cos\theta\sin\phi + B_z\sin\theta \\ B_\theta &= -B_x\sin\theta\cos\phi - B_y\sin\theta\sin\phi + B_z\cos\theta \\ B_\phi &= -B_x\sin\phi + B_y\cos\phi \end{aligned} \tag{6}$$

2 A RING

To simplify the problem we ask for the field for a complete ring, *i.e* $L = R2\pi$. In this case, there is no dependence on the toroidal angle ϕ and therefore, as customary, we evaluate the expressions for the fields at $\phi = 0$; hence

$$\begin{aligned}
B_x &= -\frac{\mu_o a R_o J_o}{4\pi}\int_0^{2\pi} d\,\phi' \int_0^{2\pi} d\,\theta' \frac{\sin\theta'\cos\phi'}{\sqrt{\alpha - \beta\cos\phi'}} \\
B_y &= -\frac{\mu_o a R_o J_o}{4\pi}\int_0^{2\pi} d\,\phi' \int_0^{2\pi} d\,\theta' \frac{\sin\theta'\sin\phi'}{\sqrt{\alpha - \beta\cos\phi'}} \\
B_z &= \frac{\mu_o a R_o J_o}{4\pi}\int_0^{2\pi} d\,\phi' \int_0^{2\pi} d\,\theta' \frac{\cos\theta'}{\sqrt{\alpha - \beta\cos\phi'}}
\end{aligned} \tag{7}$$

with

$$\begin{aligned}
\alpha &= 2*(R_o^2 + R_o r\cos\theta + R_o a\cos\theta' - ra\sin\theta\sin\theta') + r^2 + a^2 \\
\beta &= 2*(R_o^2 + R_o r\cos\theta + R_o a\cos\theta' + ra\cos\theta\cos\theta')
\end{aligned} \tag{8}$$

and also notice that $B_y = 0$.

The toroidal components of $\mathbf{B}$ are

$$\begin{aligned}
B_r &= B_x\cos\theta + B_z\sin\theta \\
B_\theta &= -B_x\sin\theta + B_z\cos\theta \\
B_\phi &= 0
\end{aligned} \tag{9}$$

A simple fortran program has been written to compute the 2D integrals; the results are shown in Figs. 1 and 2.

These results are in agreement with those found by Fernow [4].

3 Acknowledgments

I would like to thank R. Fernow for his comments

References

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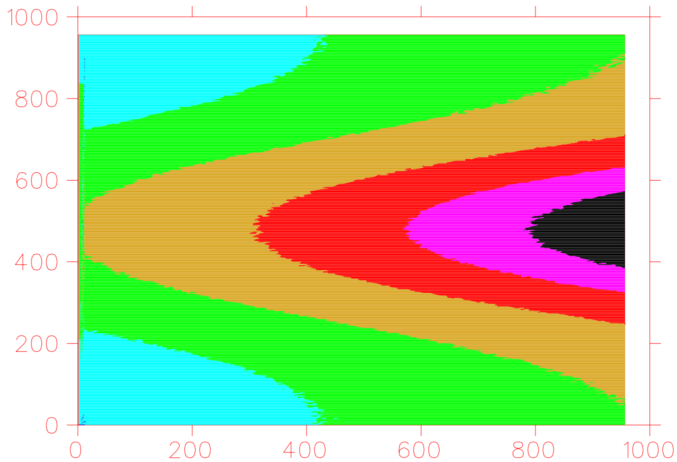

Figure 1: (Color)Plot of B_θ for $R_o = 0.20\ m$, $a = 0.10$ m and the total current is $I = 750$ kA. The horizontal axis is r in units of 10/960 cm/div and the vertical is the poloidal angle θ. in units of $\frac{2\pi}{960}$ rad/div.

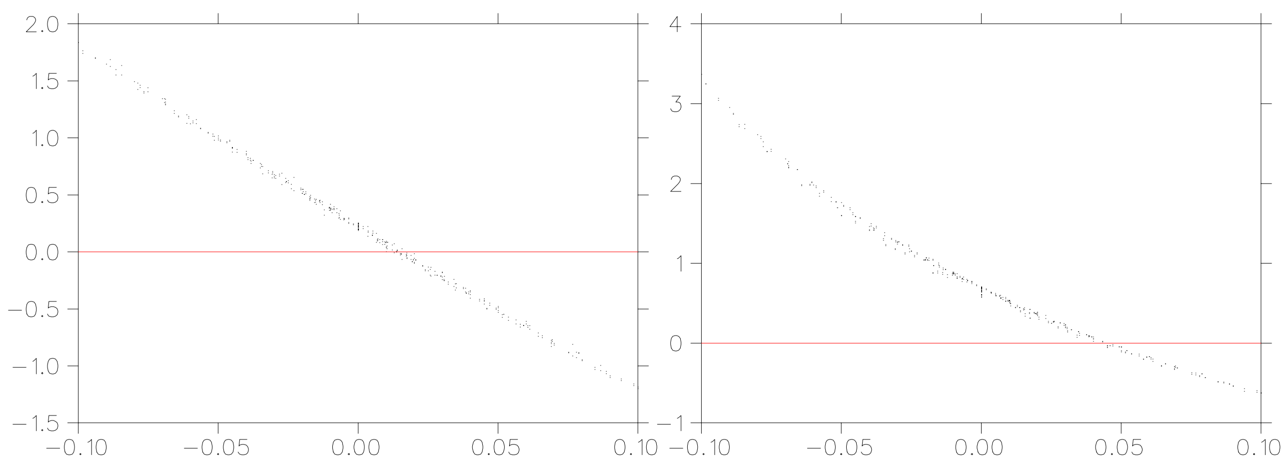

Figure 2: (Color) B_z *vs.* x for $R_o = 1.20\ m$ (left) and $R_o = 0.20\ m$ (right); in both cases the radius of the lithium lens is $a = 0.10$ m and the total current is $I = 750$ kA

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