Note on Magnetic Field Expansion with Torsion

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Abstract

We give the expansion of the magnetic field, to third order, around a helical trajectory. The explicit expressions are suitable for the case of constant curvature and torsion.

I. INTRODUCTION

We first recall the Frenet-Serret equations for a general curvilinear coordinate system (x,y,s). In a Cartesian LAB-frame of reference (X,Y,Z), the reference trajectory in our problem is given parametrically by the vector $\vec{z_o}(s)$ and for any other particle trajectory close to the reference one by

$$\vec{z}(s) = \vec{z_o}(s) + x\vec{e_x} + y\vec{e_y} \tag{1}$$

where s is the arc length on the reference trajectory and x, y are the particle deviations from the reference path. We define $\vec{e_s} = \frac{\partial \vec{z}}{\partial s} \equiv \frac{\partial \vec{z_o}}{\partial s}$, $\vec{e_x} = \frac{\partial \vec{z}}{\partial x}$, $\vec{e_y} = \frac{\partial \vec{z}}{\partial y}$. The Frenet-Serret equations read,

$$\begin{aligned} \frac{d\vec{e_x}}{ds} &= \tau\vec{e_y} + h\vec{e_s} \\ \frac{d\vec{e_y}}{ds} &= -\tau\vec{e_x} + g\vec{e_s} \\ \frac{d\vec{e_s}}{ds} &= -h\vec{e_x} - g\vec{e_y} \end{aligned} \tag{2}$$

τ is the torsion of the reference path and h g are the curvature in the horizontal and vertical planes respectively.

The covariant metric tensor is defined as

$$g_{ik} = \vec{e_i}.\vec{e_k} = \begin{pmatrix} 1 & 0 & -\tau y \\ 0 & 1 & \tau x \\ -\tau y & \tau x & H^2 + \tau^2(x^2+y^2) \end{pmatrix} \tag{3}$$

with $H = 1 + hx + gy$. The arc length $d\sigma = g_{ik}dx^i dx^k = dx^2 - 2\tau y\, dx\, ds + dy^2 + 2\tau x\, dy\, ds + (H^2 + \tau^2(x^2+y^2))ds^2$, notice that the system is not orthogonal. Next we have to calculate the contravariant tensor g^{ik} satisfying the condition $g_{ik}g^{kj} = \delta_i^j$. The expression of g^{ik} is

$$g^{ik} = \begin{pmatrix} H^2 + \tau^2 y^2 & -\tau^2 xy & \tau y \\ -\tau^2 xy & H^2 + \tau^2 x^2 & -\tau x \\ \tau y & -\tau x & 1 \end{pmatrix} \frac{1}{H^2} \tag{4}$$

Notice that the $Det(g^{ik}) = H^2$.

II. CYLINDRICAL HELIX

A helix is defined by the equations

$$\begin{aligned} X(s) &= R\cos\left(\frac{s}{\sqrt{R^2+b^2}}\right) \\ Y(s) &= R\sin\left(\frac{s}{\sqrt{R^2+b^2}}\right) \\ Z(s) &= \frac{bs}{\sqrt{R^2+b^2}} \end{aligned} \tag{5}$$

where $X(s), Y(s), Z(s)$ are cartesian coordinates in the LAB-frame of reference and R is the radius and b the period of the helical reference trajectory.

Using the notation $z_o(s) = X(s)\vec{i} + Y(s)\vec{j} + Z(s)\vec{k}$ and the definition of the tangential unit vector $\vec{e_s}$ given before, we obtain

$$\vec{e_s} = -\sin\theta\sin\frac{s}{\sqrt{R^2+b^2}}\vec{i} + \sin\theta\cos\frac{s}{\sqrt{R^2+b^2}}\vec{j} + \cos\theta\vec{k} \tag{6}$$

recalling that $\vec{e_x}$ and $\vec{e_y}$ are perpendicular to each other, we conclude

$$\begin{aligned} \vec{e_x} &= -\cos\frac{s}{\sqrt{R^2+b^2}}\vec{i} - \sin\frac{s}{\sqrt{R^2+b^2}}\vec{j} \\ \vec{e_y} &= \cos\theta\sin\frac{s}{\sqrt{R^2+b^2}}\vec{i} - \cos\theta\cos\frac{s}{\sqrt{R^2+b^2}}\vec{j} + \sin\theta\vec{k} \end{aligned} \tag{7}$$

Using the Frenet-Serret equations and recalling that $g = 0$ for a helical trajectory, then $h = -\frac{R}{R^2+b^2}$ $\quad \tau = \frac{b}{R^2+b^2}$ for the curvature and torsion respectively. Furthermore, we can write the relation between field components (we denote B_i the cartesian components in the LAB-frame)

$$\begin{aligned} B_x &= -B_1\cos\frac{s}{\sqrt{R^2+b^2}} - B_2\sin\frac{s}{\sqrt{R^2+b^2}} \\ B_y &= B_1\cos\theta\,\sin\frac{s}{\sqrt{R^2+b^2}} - B_2\cos\theta\,\cos\frac{s}{\sqrt{R^2+b^2}} + B_3\sin\theta \\ B_s &= -B_1\sin\theta\,\sin\frac{s}{\sqrt{R^2+b^2}} + B_2\sin\theta\cos\frac{s}{\sqrt{R^2+b^2}} + B_3\cos\theta \end{aligned} \tag{8}$$

III. MAGNETIC FIELD

From the Lorentz equation $\frac{d\vec{p}}{dt} = q\left(\vec{v}\times\vec{B}\right)$ is not hard to shown that

$$\begin{aligned} h &= -\frac{q}{\gamma m|v|}B_y(0,0,s) \\ g &= \frac{q}{\gamma m|v|}B_x(0,0,s) \\ \tau &= \frac{q}{\gamma m|v|}B_s(0,0,s) \end{aligned} \tag{9}$$

As customary we assume an scalar magnetic potential of the form $\Phi(x,y,s) = \sum_{n,m} a_{n,m}(s)\frac{x^n}{n!}\frac{y^m}{m!}$. In this general coordinate system

$$\vec{B} = -\mathbf{grad}(\Phi) = -g^{ik}\frac{\partial B^i}{\partial x^k} \tag{10}$$

$$\mathbf{div}(\vec{B}) = \frac{1}{H}\left(\frac{\partial HB_x}{\partial x} + \frac{\partial HB_y}{\partial y} + \frac{\partial HB_s}{\partial s}\right) = 0 \tag{11}$$

The explicit expression of the magnetic field components are:

$$B_x(x,y,s) = \frac{H^2+\tau^2y^2}{H^2}\frac{\partial\Phi}{\partial x} - \frac{\tau^2yx}{H^2}\frac{\partial\Phi}{\partial y} + \frac{\tau y}{H^2}\frac{\partial\Phi}{\partial s} \tag{12}$$

$$B_y(x,y,s) = -\frac{\tau^2yx}{H^2}\frac{\partial\Phi}{\partial x} + \frac{H^2+\tau^2x^2}{H^2}\frac{\partial\Phi}{\partial y} - \frac{\tau x}{H^2}\frac{\partial\Phi}{\partial s} \tag{13}$$

$$B_s(x,y,s) = \frac{\tau y}{H^2}\frac{\partial\Phi}{\partial x} - \frac{\tau x}{H^2}\frac{\partial\Phi}{\partial y} + \frac{1}{H^2}\frac{\partial\Phi}{\partial s} \tag{14}$$

Using the same notation as in [1], in third order, the magnetic field is given by (we use the notation $\frac{da}{ds} \equiv \dot{a}$)

$$\begin{aligned} &B_x(x,y,s) \approx a_{10} + xa_{20} + y\left[a_{11}+\tau\dot{a}_{00}\right] + xy\left[a_{21}-\tau^2a_{01}+\tau\dot{a}_{10}-2\tau h\dot{a}_{00}\right] + \frac{1}{2}x^2a_{30} + \\ &y^2\left[\frac{1}{2}a_{12}+\tau^2a_{10}+\tau\dot{a}_{01}-2\tau g\dot{a}_{00}\right] + \frac{1}{6}x^3a_{40} + \\ &x^2y\left[\frac{1}{2}a_{31}+3\tau h^2\dot{a}_{00}-\tau^2a_{11}+2\tau^2ha_{01}+\frac{1}{2}\tau\dot{a}_{20}-2\tau h\dot{a}_{10}\right] + \\ &xy^2\left[\frac{1}{2}a_{22}+\tau^2a_{20}-2\tau^2ha_{10}-\tau^2a_{02}+2\tau^2ga_{01}+\tau\dot{a}_{11}-2\tau g\dot{a}_{10}-2\tau h\dot{a}_{01}+6\tau hg\dot{a}_{00}\right] + \\ &y^3\left[\frac{1}{6}a_{13}+\tau^2a_{11}-2\tau^2ga_{10}+\frac{1}{2}\tau\dot{a}_{02}-2\tau g\dot{a}_{01}+3\tau g^2\dot{a}_{00}\right] \end{aligned} \tag{15}$$

$$
\begin{aligned}
&B_y(x,y,s) \approx a_{01} + y a_{02} + x\left[a_{11} - \tau \dot{a}_{00}\right] + xy\left[a_{12} - \tau^2 a_{10} - \tau \dot{a}_{01} + 2\tau g \dot{a}_{00}\right] + \frac{1}{2} y^2 a_{03} + \\
&x^2\left[\frac{1}{2} a_{21} + \tau^2 a_{01} - \tau \dot{a}_{10} + 2\tau h \dot{a}_{00}\right] + \frac{1}{6} y^3 a_{04} + \\
&xy^2\left[\frac{1}{2} a_{13} - 3\tau g^2 \dot{a}_{00} - \tau^2 a_{11} + 2\tau^2 g a_{10} - \frac{1}{2}\tau \dot{a}_{02} + 2\tau g \dot{a}_{01}\right] + \\
&x^2 y\left[\frac{1}{2} a_{22} - \tau^2 a_{20} + 2\tau^2 h a_{10} + \tau^2 a_{02} - 2\tau^2 g a_{01} - \tau \dot{a}_{11} + 2\tau h \dot{a}_{01} + 2\tau g \dot{a}_{10} - 6\tau h g \dot{a}_{00}\right] + \\
&x^3\left[\frac{1}{6} a_{31} + \tau^2 a_{11} - 2\tau^2 h a_{01} - \frac{1}{2}\tau \dot{a}_{20} + 2\tau h \dot{a}_{10} - 3\tau h^2 \dot{a}_{00}\right]
\end{aligned} \tag{16}
$$

$$
\begin{aligned}
&B_s(x,y,s) \approx \dot{a}_{00} + x\left[-\tau a_{01} + \dot{a}_{10} - 2h\dot{a}_{00}\right] + y\left[\tau a_{10} + \dot{a}_{01} - 2g\dot{a}_{00}\right] + \\
&x^2\left[-\tau a_{11} + 2\tau h a_{01} + \frac{1}{2}\dot{a}_{20} - 2h\dot{a}_{10} + 3h^2\dot{a}_{00}\right] + y^2\left[\tau a_{11} - 2\tau g a_{10} + \frac{1}{2}\dot{a}_{02} - 2g\dot{a}_{01} + \right. \\
&\left. 3g^2\dot{a}_{00}\right] + xy\left[\tau a_{20} - 2\tau h a_{10} - \tau a_{02} + 2\tau g a_{01} + \dot{a}_{11} - 2h\dot{a}_{01} - 2g\dot{a}_{10} + 6hg\dot{a}_{00}\right] + \\
&x^3\left[-\frac{1}{2}\tau a_{21} + 2\tau h a_{11} - 3\tau h^2 a_{01} + \frac{1}{6}\dot{a}_{30} - h\dot{a}_{20} + 3h^2\dot{a}_{10} - 4h^3\dot{a}_{00}\right] + \\
&y^3\left[\frac{1}{2}\tau a_{12} - 2\tau g a_{11} + 3\tau g^2 a_{10} + \frac{1}{6}\dot{a}_{03} - g\dot{a}_{02} + 3g^2\dot{a}_{01} - 4g^3\dot{a}_{00}\right] + \\
&x^2 y\left[\frac{1}{2}\tau a_{30} - 2\tau h a_{20} + 3\tau h^2 a_{10} - \tau a_{12} + 2\tau h a_{02} + 2\tau g a_{11} - 6\tau h g a_{01} + \frac{1}{2}\dot{a}_{21} - 2h\dot{a}_{11} - \right. \\
&\left. g\dot{a}_{20} + 3h^2\dot{a}_{01} + 6hg\dot{a}_{10} - 12h^2 g\dot{a}_{00}\right] + \\
&xy^2\left[-\frac{1}{2}\tau a_{03} + 2\tau g a_{02} - 3\tau g^2 a_{01} + \tau a_{21} - 2\tau g a_{20} - 2\tau h a_{11} + 6\tau h g a_{10} + \frac{1}{2}\dot{a}_{12} - 2g\dot{a}_{11} - \right. \\
&\left. h\dot{a}_{02} + 3g^2\dot{a}_{10} + 6hg\dot{a}_{01} - 12g^2 h\dot{a}_{00}\right]
\end{aligned} \tag{17}
$$

Following references [1] and [2] we assume we know, measured, the field components in the horizontal plane, *i.e* $B_x(x,0,s) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + ...$ $B_y(x,0,s) = b_0 + b_1 x + b_2 x^2 + b_3 x^3 + ...$ where we have used the notation $a_{n-1} = (n-1)! a_{n0} \qquad b_n = n! a_{n1}$, for the normal and skew multipoles, respectively. Eq. 11 is equivalent to the following conditions for the 6

unknowns up to third order in the magnetic field $(a_{02}, a_{12}, a_{03}, a_{22}, a_{04}, a_{13})$

$$
\begin{aligned}
a_{02} + a_{20} &= -\ddot{a}_{00} - ha_{10} - ga_{01} \\
a_{12} + a_{30} &= -2ha_{20} - ga_{11} + \tau^2 a_{10} - \tau g\dot{a}_{00} - \ddot{a}_{10} + h\ddot{a}_{00} - ha_{02} + 2\tau\dot{a}_{01} \\
a_{03} + a_{21} &= -ha_{11} + \tau h\dot{a}_{00} - 2ga_{02} - ga_{20} + \tau^2 a_{01} - 2\tau\dot{a}_{10} + g\ddot{a}_{00} - \ddot{a}_{01} \\
a_{22} + a_{40} &= -3ha_{30} - ga_{21} + 2\tau^2 ga_{01} - 2\tau g\dot{a}_{10} - 2\tau^2 ha_{10} - 2ha_{12} - 4\tau h\dot{a}_{01} + \\
&\quad 2\tau^2 a_{20} - 2\tau^2 a_{02} - 4\tau\dot{a}_{11} + 4\tau hg\dot{a}_{00} + 2h\ddot{a}_{10} - 2h^2\ddot{a}_{00} - \ddot{a}_{20} \\
a_{04} + a_{22} &= -ha_{12} - 2ga_{21} + 2\tau^2 ha_{10} - 3ga_{03} - 2\tau^2 a_{20} + 2\tau^2 a_{02} + 2\tau h\dot{a}_{01} - 2\tau^2 ga_{01} - \\
&\quad 4\tau hg\dot{a}_{00} - 4\tau\dot{a}_{11} + 4\tau g\dot{a}_{10} - \ddot{a}_{02} + 2g\ddot{a}_{01} - 2g^2\ddot{a}_{00} \\
a_{13} + a_{31} &= -2ha_{21} - 2ga_{12} - 2\tau^2 ha_{01} - 2\tau^2 ga_{10} + 3\tau h\dot{a}_{10} - 3\tau g\dot{a}_{01} - \\
&\quad 2\tau h^2\dot{a}_{00} + 2\tau g^2\dot{a}_{00} + 4\tau^2 a_{11} - 2\tau\dot{a}_{20} + 2\tau\dot{a}_{02} - ha_{03} - ga_{30} + \\
&\quad h\ddot{a}_{01} + g\ddot{a}_{10} - 2hg\ddot{a}_{00} - \ddot{a}_{11}
\end{aligned}
$$

The fields are identical to those in reference [1] (notice minor typos) after setting $a_{10} = 0$ and $g = \tau = 0$.

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[1] Chun-xi Wang and Lee C. Teng, *Magnetic Field Expansion for Particle Tracking in a Bent-Solenoid Channel*, NFMCC-doc-201-v1 **http://nfmcc-docdb.fnal.gov/cgi-bin/ShowDocument?docid=207**

[2] Andreas C. Kabel, *Maxwell-Lorentz Equations in general Frenet-Serret coordinates* PAC03

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