Twiss parameters and beam matrix formulation of generalized Courant-Snyder theory for coupled transverse dynamics and their application to helical transport channels

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Abstract

In recent papers [1, 2] Qin and Davidson have generalized Courant-Snyder (CS) theory for one degree of freedom to the case of coupled transverse dynamics with two degrees of freedom. The generalized theory has four basic components of the original CS theory, i.e., the envelope equation, phase advance, transfer matrix, and the CS invariant, all of which have their counterparts in the original CS theory with remarkably similar expressions. In this paper, we further investigate this remarkable similarity between the original and generalized CS theory, and construct the Twiss parameters and beam matrix in generalized form, which can be used to provide a practical framework for accelerator design, transverse beam measurement and control, and particle tracking studies. In particular, it is shown that choosing the appropriate initial conditions for the matrix envelope equation is important to be consistent with the symplectic condition of the transfer matrix, and to simplify the calculation of the beam matrix. As an illustrative example, the generalized form of the Twiss parameters and beam matrix has been applied to the case of a helical transport channel, where the two transverse motions are strongly coupled.

I. INTRODUCTION

In recent papers [1, 2] Qin and Davidson generalized the Courant-Snyder theory [3–6] for one degree of freedom to the case of coupled transverse dynamics with two degrees of freedom using a time-dependent canonical transformation technique. Although there are several alternative parametrization methods for coupled transverse dynamics, such as the Teng-Edward parametrization [7, 8], the Mais-Ripken parametrization [9–11], the normal form method [12], and the SLIM formalism [13], the Qin-Davidson parametrization is noteworthy in the sense that it retains four basic components of the original Courant-Snyder theory, i.e., the envelope equation, phase advance, transfer matrix, and the Courant-Snyder invariant, with remarkably similar expressions to their counterparts in the original Courant-Snyder theory. This feature provides a formulation closer in structure to the original Courant-Snyder theory, and enables one to deal with more complicated coupled dynamics in the context of the well-established Courant-Snyder formalism. In this paper, we further investigate this remarkable similarity between the original and generalized Courant-Snyder theory, and construct the Twiss parameters (α, β, and γ) and beam matrix (Σ) in generalized forms, which can provide a practical framework for accelerator design, transverse beam measurement and control, and particle tracking studies.

The organization of this paper is the following. In Sec. II, we introduce the generalized Courant-Snyder theory based on Refs. [1] and [2]. The Twiss parameters and beam matrix are formulated in generalized forms in Secs. III and IV, respectively. In Sec. V, we discuss the uniqueness of the matched solutions to the matrix envelope equation. Finally, a numerical example of the calculation of the Twiss parameters and beam matrix is given in Sec. VI for the case of a helical transport channel.

II. GENERALIZED COURANT-SNYDER THEORY

The general form of the Hamiltonian for the coupled transverse dynamics is given by

$$H_c = \frac{1}{2} u^T A_c u, \tag{1}$$

where

$$A_c = \begin{pmatrix} \kappa & R \\ R^T & I \end{pmatrix}, \tag{2}$$

$$u = (x, y, p_x, p_y)^T, \tag{3}$$

$$\kappa(s) = \begin{pmatrix} \kappa_x & \kappa_{xy} \\ \kappa_{xy} & \kappa_y \end{pmatrix}. \tag{4}$$

Here, the 2×2 matrix $\kappa(s)$ is time-dependent and symmetric ($\kappa = \kappa^T$), R is an arbitrary, time-dependent 2×2 matrix, and I is the 2×2 unit matrix. The variable s is the path length that plays the role of a time-like variable. The superscript "T" denotes the transpose operation of a matrix, and $p_x(p_y)$ is the scaled canonical momentum variable conjugate to the transverse coordinate $x(y)$ relative to the reference orbit. For a combination of all the *linear* components of a focusing lattice, i.e., the dipole, quadrupole, skew quadrupole, and solenoidal components, we find [2, 11, 14]

$$\kappa(s) = \begin{pmatrix} \Omega^2 + \kappa_q + \frac{1}{\rho_x^2} & \kappa_{sq} + \frac{1}{\rho_x \rho_y} \\ \kappa_{sq} + \frac{1}{\rho_x \rho_y} & \Omega^2 - \kappa_q + \frac{1}{\rho_y^2} \end{pmatrix}, \tag{5}$$

$$R(s) = \begin{pmatrix} 0 & -\Omega \\ +\Omega & 0 \end{pmatrix}, \tag{6}$$

where κ_q is the quadrupole focusing coefficient, Ω is one-half of the normalized relativistic Larmor frequency associated with the solenoidal lattice [15], κ_{sq} is the skew quadrupole coefficient, and $\rho_x(\rho_y)$ is the bending radius in the $x(y)$-direction associated with the dipole field. Note that all of the elements in the matrices $\kappa(s)$ and $R(s)$ are generally time-dependent.

If we apply the final results of the generalized Courant-Snyder theory obtained in Refs. [1] and [2] to the Hamiltonian in Eq. (1), we can express the solution for the transverse dynamics in terms of a time-dependent linear map from the initial condition u_0, i.e.,

$$u(s) = M_c u_0, \tag{7}$$

where the transfer matrix M_c is given by

$$M_c = Q^{-1} S^{-1} P^{-1} S_0, \tag{8}$$

$$Q^{-1} = \begin{pmatrix} Q_4^T & 0 \\ 0 & Q_4^T \end{pmatrix}, \tag{9}$$

$$Q_4 = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}, \tag{10}$$

$$\theta' = \Omega, \tag{11}$$

$$S^{-1} = \begin{pmatrix} w^T & 0 \\ w^{-1}w'w^T & w^{-1} \end{pmatrix}, \tag{12}$$

$$P^{-1} = \begin{pmatrix} P_1 & -P_2 \\ P_2 & P_1 \end{pmatrix}, \tag{13}$$

$$S_0 = \begin{pmatrix} (w_0^{-1})^T & 0 \\ -w_0' & w_0 \end{pmatrix}. \tag{14}$$

Here, w is the 2×2 envelope matrix satisfying the following non-commutative matrix envelope equation [1, 2]

$$w'' + w\tilde{\kappa} = (w^{-1})^T w^{-1} (w^{-1})^T, \tag{15}$$

with

$$\tilde{\kappa} = Q_4 \kappa Q_4^{-1}, \tag{16}$$

and (w_0, w_0') denotes the initial conditions for w and w'. The prime denotes a derivative with respect to s. The rotation matrix P^{-1} is determined from the generalized phase advance equations

$$P_1' = P_2 \beta_I, \tag{17}$$

$$P_2' = -P_1 \beta_I, \tag{18}$$

where the matrix phase advance rate β_I is

$$\beta_I = (ww^T)^{-1}. \tag{19}$$

The generalized Courant-Snyder invariant is

$$I_c^{CS} = u^T Q^T S^T S Q u. \tag{20}$$

Derivation of these results using a time-dependent canonical transformation technique is described in more detail in Refs. [1] and [2]. These results are the non-commutative generalization of the Courant-Snyder theory for one degree of freedom to the case of coupled transverse dynamics with two degrees of freedom.

III. TWISS PARAMETERS

In the original Courant-Snyder theory [3], the Twiss parameters α, β, and γ were introduced and provided an important formulation to describe the evolution of the beam distribution in trace space [16]. Therefore, in this section, we generalize the Twiss parameters to the case of coupled transverse dynamics with two degrees of freedom using the matrix envelope equation (15) and the generalized Courant-Snyder invariant defined in Eq. (20). Using the fact that $\tilde{\kappa}$ is symmetric, we can rewrite the matrix envelope equation (15) in two parts:

$$(w^T w)'' + \tilde{\kappa}(w^T w) + (w^T w)\tilde{\kappa} = 2\left[(w^T w)^{-1} + w^{T'} w'\right], \tag{21}$$

$$w'' w^T = w w^{T''}. \tag{22}$$

To obtain Eq. (21), we operate on Eq. (15) with $w^T(\cdots) + (\cdots)^T w$. Similarly, Eq. (22) is derived after operating on Eq. (15) with $(\cdots)w^T - w(\cdots)^T$. Due to the symmetric property of the matrix equations, Eq. (21) gives three independent coupled differential equations, while Eq. (22) gives only one. On the other hand, from the generalized form of the Courant-Snyder invariant in Eq. (20), we note that the trace-space ellipse is determined in the Larmor frame [15] by the matrix

$$\begin{aligned} S^T S &= \begin{pmatrix} w^{-1} & -w^{T'} \\ 0 & w^T \end{pmatrix} \begin{pmatrix} (w^{-1})^T & 0 \\ -w' & w \end{pmatrix} \\ &= \begin{pmatrix} (w^T w)^{-1} + w^{T'} w' & -w^{T'} w \\ -w^T w' & w^T w \end{pmatrix}. \end{aligned} \tag{23}$$

Comparing Eqs. (21) and (23), we define the generalized Twiss parameters as follows:

$$\alpha = -w^T w', \tag{24}$$

$$\beta = w^T w, \tag{25}$$

$$\gamma = (w^T w)^{-1} + w^{T'} w'. \tag{26}$$

Here, the generalized Twiss parameters α, β, and γ are 2×2 matrixes, and $\beta = \beta^T$ and $\gamma = \gamma^T$, while $\alpha \neq \alpha^T$ in general. The differential equation for the beta-function matrix β becomes

$$\beta'' + \left[(\tilde{\kappa}\beta) + (\tilde{\kappa}\beta)^T\right] = 2\gamma, \tag{27}$$

and the derivative of β yields

$$\beta' = (w^T w)' = w^{T'} w + w^T w' = -(\alpha + \alpha^T), \tag{28}$$

both of which are non-commutative generalizations of their counterparts in the original Courant-Snyder theory with remarkably similar expressions. Here, we define $\beta = w^T w$, which is different from the definition in Refs. [1] and [2], where β is defined as $\beta = ww^T = \beta_I^{-1}$.

Equation (22) also provides very valuable information. Integration by parts of Eq. (22) yields

$$w' w^T - w w^{T'} = const. \times \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \tag{29}$$

where the integration constant is arbitrary, and will be determined from the initial conditions $(w, w')_0 = (w_0, w_0')$. Since the single-particle trajectory following Eq. (7) is indeed independent of the choice of $(w, w')_0$, we have the freedom to choose the integration constant. Here, we demonstrate that we should choose $const. = 0$ to be consistent with the fact that the time-dependent symplectic matrix S in Eq. (12) gives the canonical transformation [2]. From the differential equation that S satisfies, which is derived in Ref. [17], we find

$$\beta_I' w + 2\beta_I w' = 0. \tag{30}$$

Furthermore, the time derivative of the definition of the matrix phase advance rate, $\beta_I w w^T = I$, leads to

$$\beta_I' w = -\beta_I (w w^T)' (w^T)^{-1} = -\beta_I \left[w' + w (w^{-1} w')^T \right]. \tag{31}$$

Combining Eqs. (30) and (31), we obtain

$$w' w^T = w w^{T'}, \tag{32}$$

which gives $const. = 0$ in Eq. (29). It should be noted that Eq. (32) gives only one independent differential equation [see the explicit expression in Eq. (48) of Sec. V]. Equation (32) makes the expression for S^{-1} much simpler, i.e.,

$$S^{-1} = \begin{pmatrix} w^T & 0 \\ w^{-1} w' w^T & w^{-1} \end{pmatrix} = \begin{pmatrix} w^T & 0 \\ w^{T'} & w^{-1} \end{pmatrix}, \tag{33}$$

and readily gives the matrix version of the familiar relation between α, β, and γ, i.e.,

$$\beta\gamma = I + w^T w w^{T'} w' = I + \alpha^2. \tag{34}$$

IV. BEAM MATRIX

To describe the beam distribution in the four-dimensional trace space (x, y, x', y'), we consider a multivariate Gaussian in the following form

$$f(X) = N \exp\left[-\frac{1}{2}X^T\Sigma^{-1}X\right], \tag{35}$$

where $\Sigma = \langle XX^T \rangle$ is the covariant matrix which will turn out to be the beam matrix, and $N = (2\pi)^{-2}[\det(\Sigma)]^{-1/2}$ is a normalization constant. For simplicity, we define $X = (x, y, x', y')^T$, and assume $\langle X \rangle = 0$ (i.e., any centroid offset is disregarded, or the coordinates are redefined with respect to the offset [18]). Because the transfer matrix introduced in Eq. (7) is for the canonical variables $u = (x, y, p_x, p_y)^T$, we need to transform these variables to trace-space (geometrical) variables $X = (x, y, x', y')^T$, in which the beam distribution is usually described (particulary for experimental measurements) [19]. For this purpose, we introduce the following matrix

$$U(s) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & -\Omega & 1 & 0 \\ +\Omega & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} I & 0 \\ R & I \end{pmatrix}, \tag{36}$$

which gives $u = UX$. Note that $\det(U) = \det(U^T) = \det(U^{-1}) = 1$, while $U^T \neq U^{-1}$ in general. From the generalized Courant-Snyder invariant in Eq. (20), and the definitions of the generalized Twiss parameters in the previous section, we find

$$I_c^{CS} = X^T U^T Q^T \begin{pmatrix} \gamma & \alpha^T \\ \alpha & \beta \end{pmatrix} QUX. \tag{37}$$

For the Gaussian beam distribution considered here, the 4D rms trace-space ellipse can be determined by the $\exp[-1/2]$ contour of the distribution function $f(X)$ [16]. Therefore, after setting $I_c^{CS} = \sqrt{\epsilon_{4\text{D}}}$ (which makes $\epsilon_{4\text{D}} = \sqrt{\det(\Sigma)}$ as in the usual convention), we find

$$\begin{aligned} 1 &= X^T\Sigma^{-1}X \\ &= \frac{1}{\sqrt{\epsilon_{4\text{D}}}} X^T U^T Q^T \begin{pmatrix} \gamma & \alpha^T \\ \alpha & \beta \end{pmatrix} QUX. \end{aligned} \tag{38}$$

Furthermore, we obtain the following expression for the beam matrix

$$\Sigma = \sqrt{\epsilon_{4\text{D}}} \times U^{-1}Q^T \begin{pmatrix} \gamma & \alpha^T \\ \alpha & \beta \end{pmatrix}^{-1} Q(U^T)^{-1}. \tag{39}$$

Using the property of the symplectic matrix, $\det\left[(S^T S)^{-1}\right] = 1$, we can readily show that $\det(\Sigma) = \epsilon_{4\mathrm{D}}^2$, as expected, and the volume enclosed by a 4D rms trace-space ellipse is $V_{4\mathrm{D}} = (\pi^2/2)\sqrt{\det(\Sigma)} = (\pi^2/2)\epsilon_{4\mathrm{D}}$. If we apply Eq. (33), or equivalently Eq. (32), we can further simplify the expression for the beam matrix. Because

$$\begin{aligned}
\begin{pmatrix} \gamma & \alpha^T \\ \alpha & \beta \end{pmatrix}^{-1} &= S^{-1}(S^{-1})^T \\
&= \begin{pmatrix} w^T w & w^T w' \\ w^{T'} w & (w^T w)^{-1} + w^{T'} w' \end{pmatrix} \\
&= \begin{pmatrix} \beta & -\alpha \\ -\alpha^T & \gamma \end{pmatrix}, \qquad (40)
\end{aligned}$$

we obtain a remarkably similar expression for the beam matrix as in the the original Courant-Snyder theory. Note that Eq. (40) is valid because $w' w^T = w w^{T'}$. Finally, we assemble all of the calculations together into the following explicit form:

$$\begin{aligned}
\Sigma &= \begin{pmatrix} \langle x^2\rangle & \langle xy\rangle & \langle xx'\rangle & \langle xy'\rangle \\ \langle yx\rangle & \langle y^2\rangle & \langle yx'\rangle & \langle yy'\rangle \\ \langle x'x\rangle & \langle x'y\rangle & \langle x'^2\rangle & \langle x'y'\rangle \\ \langle y'x\rangle & \langle y'y\rangle & \langle y'x'\rangle & \langle y'^2\rangle \end{pmatrix} \\
&= \sqrt{\epsilon_{4\mathrm{D}}} \times \begin{pmatrix} I & 0 \\ -R & I \end{pmatrix} \begin{pmatrix} Q_4^T & 0 \\ 0 & Q_4^T \end{pmatrix} \begin{pmatrix} \beta & -\alpha \\ -\alpha^T & \gamma \end{pmatrix} \begin{pmatrix} Q_4 & 0 \\ 0 & Q_4 \end{pmatrix} \begin{pmatrix} I & -R^T \\ 0 & I \end{pmatrix}. \qquad (41)
\end{aligned}$$

Here, we note that $\Sigma = \Sigma^T$.

As an illustrative example of the calculation of the beam matrix using Eq. (41), we consider one of the two-dimensional subspaces of the four-dimensional trace space. Particularly for beam profile measurements, the (x, y)-plane is the most obvious projection which shows the beam cross section under the influence of the coupling [14]. In general, the beam cross section becomes tilted due to the coupling, with tilt angle ξ given by

$$\tan 2\xi = \frac{2\langle xy\rangle}{\langle x^2\rangle - \langle y^2\rangle}. \qquad (42)$$

The tilt angle is generally time-dependent (i.e., varies along the beam transport line), and is not well-defined when $\langle x^2\rangle = \langle y^2\rangle$ [14]. According to Eq (41), we can express $\langle x^2\rangle$, $\langle y^2\rangle$, and $\langle xy\rangle$ in terms of the elements of β or w as follows:

$$\begin{aligned}\langle x^2\rangle &= \sqrt{\epsilon_{\rm 4D}}(Q_4^T\beta Q_4)_{11}\\ &= \sqrt{\epsilon_{\rm 4D}}\left(\beta_{11}\cos^2\theta+\beta_{12}\cos\theta\sin\theta+\beta_{21}\sin\theta\cos\theta+\beta_{22}\sin^2\theta\right)\\ &= \sqrt{\epsilon_{\rm 4D}}\left[(w_1^2+w_3^2)\cos^2\theta+2(w_1w_2+w_3w_4)\cos\theta\sin\theta+(w_2^2+w_4^2)\sin^2\theta\right],\end{aligned} \tag{43}$$

$$\begin{aligned}\langle y^2\rangle &= \sqrt{\epsilon_{\rm 4D}}(Q_4^T\beta Q_4)_{22}\\ &= \sqrt{\epsilon_{\rm 4D}}\left(\beta_{11}\sin^2\theta-\beta_{12}\sin\theta\cos\theta-\beta_{21}\cos\theta\sin\theta+\beta_{22}\cos^2\theta\right)\\ &= \sqrt{\epsilon_{\rm 4D}}\left[(w_1^2+w_3^2)\sin^2\theta-2(w_1w_2+w_3w_4)\sin\theta\cos\theta+(w_2^2+w_4^2)\cos^2\theta\right],\end{aligned} \tag{44}$$

$$\begin{aligned}\langle xy\rangle &= \sqrt{\epsilon_{\rm 4D}}(Q_4^T\beta Q_4)_{12}=\sqrt{\epsilon_{\rm 4D}}(Q_4^T\beta Q_4)_{21}\\ &= \sqrt{\epsilon_{\rm 4D}}\left(-\beta_{11}\cos\theta\sin\theta+\beta_{12}\cos^2\theta-\beta_{21}\sin^2\theta+\beta_{22}\sin\theta\cos\theta\right)\\ &= \sqrt{\epsilon_{\rm 4D}}\left[(w_2^2+w_4^2-w_1^2-w_3^2)\cos\theta\sin\theta+(w_1w_2+w_3w_4)(\cos^2\theta-\sin^2\theta)\right].\end{aligned} \tag{45}$$

Here, w_1, w_2, w_3, and w_4 are the four elements of w, i.e.,

$$w=\begin{pmatrix} w_1 & w_2\\ w_3 & w_4\end{pmatrix}. \tag{46}$$

Finally, we note that the rms beam radius R_b can be expressed as

$$R_b^2=\langle x^2+y^2\rangle=\sqrt{\epsilon_{\rm 4D}}(\beta_{11}+\beta_{22})=\sqrt{\epsilon_{\rm 4D}}(w_1^2+w_2^2+w_3^2+w_4^2), \tag{47}$$

which does not explicitly depend on the accumulated phase of rotation θ associated with solenoidal field.

V. DISCUSSION

In the pervious two sections, we have constructed the Twiss parameters and beam matrix in the context of generalized Courant-Snyder theory. Once the matrix envelope function w is known, we can effectively describe the evolution of a beam distribution in the four-dimensional trace space. To numerically integrate Eq. (15), we need to specify eight initial values, i.e., $(w_1, w_2, w_3, w_4)_0$ and $(w_1', w_2', w_3', w_4')_0$, which satisfy $w'w^T - ww^{T'} = 0$ at $s = 0$. In terms of the elements of w, this condition can be expressed as

$$(w_1'w_3+w_2'w_4-w_3'w_1-w_4'w_2)_0=0. \tag{48}$$

In a closed (or periodic) lattice system, it is desirable to find periodically matched solutions for w to construct the β functions, in which case the trace-space ellipse specified by the Courant-Snyder invariant also becomes periodic. The periodic matching conditions are

$$(w_1, w_2, w_3, w_4)_0 = (w_1, w_2, w_3, w_4)_L, \tag{49}$$

$$(w_1', w_2', w_3', w_4')_0 = (w_1', w_2', w_3', w_4')_L, \tag{50}$$

where L is the lattice periodicity length. When w is the solution of the matrix envelope equation (15), it follows automatically from Eq. (29) that

$$(w_1'w_3 + w_2'w_4 - w_3'w_1 - w_4'w_2)_0 = (w_1'w_3 + w_2'w_4 - w_3'w_1 - w_4'w_2)_L. \tag{51}$$

Hence, one of the eight constraints in Eqs. (49) and (50) is redundant, and only seven of them are indeed independent.

It is interesting to note that the matrix envelope equation (15) admits an orthogonal symmetry. Suppose that we have an arbitrary constant orthogonal matrix C, i.e., $C^TC = I$. Operating on Eq. (15) with $C(\cdots)$, and rearranging terms with $I = C^TC$, readily give

$$\begin{aligned} Cw'' + Cw\tilde{\kappa} &= C(w^{-1})^T w^{-1} C^T C (w^{-1})^T \\ &= \left[(Cw)^{-1}\right]^T (Cw)^{-1} \left[(Cw)^{-1}\right]^T. \end{aligned} \tag{52}$$

If w is the solution of the matrix envelope equation (15) with the condition in Eq. (48) and periodic boundary conditions in Eqs. (49) and (50), then it follows automatically from Eq. (52) that $\tilde{w} = Cw$ is also a solution that satisfies $(\tilde{w}'\tilde{w}^T - \tilde{w}\tilde{w}^{T'})_0 = 0$ and $(\tilde{w}, \tilde{w}')_0 = (\tilde{w}, \tilde{w}')_L$. Indeed, this multiplicity of solutions is found in the original Courant-Snyder theory as well. For example, the sign of the envelope function w is not determined from the 1D envelope equation, but only the positive solution is used in calculations for convenience [6].

On the other hand, it should be emphasized that the function $\beta = w^Tw = w^TC^TCw = \tilde{w}^T\tilde{w}$ is unique for all the solutions of w in the same orthogonal group. We note also from Eqs. (27), (28), and (34) that matched solutions for α and γ can be uniquely determined in terms of the three elements of the beta-function matrix, β_{11}, $\beta_{12}(=\beta_{21})$, and β_{22}. In Sec. VI we will demonstrate matched solutions for a non-conventional system, such as a helical transport channel.

VI. APPLICATION TO THE HELICAL TRANSPORT CHANNEL

To achieve fast and simultaneous reduction of the transverse and longitudinal phase spaces of intense muon beams for a muon collider and a neutrino factory, a cooling scheme that employs a helical magnetic field transport channel has recently been proposed [20]. The helical transport channel is composed of a solenoidal magnetic field component, which does not change direction, and transverse dipole and quadrupole magnetic field components, which change direction along the channel axis with helical symmetry. The periodic equilibrium orbit in the channel becomes a helix with the axial periodicity length λ of the helical magnetic field, and the equilibrium radius is determined by the particle momentum along the helical orbit, together with the pitch and strength of the field configuration. The linear equations of motion for coupled transverse dynamics about the equilibrium orbit can be expressed in the helically rotating frame as [21]

$$x'' + \left[\frac{1}{\rho_x^2} + \kappa_q\right] x - 2\Omega y' = 0, \tag{53}$$

$$y'' - \kappa_q y + 2\Omega x' = 0, \tag{54}$$

where (x, y) is the scaled transverse coordinate relative to the equilibrium orbit, and the coordinate along the channel z-axis has been chosen as the time-like independent variable. Since the externally-imposed helical field structure is continuous, the bending radius ρ_x, the quadrupole focusing coefficient κ_q, and the Larmor frequency Ω are all z-independent, making the dynamical system conservative. For a complete analysis of emittance exchange, energy loss by ionization, and the resultant six-dimensional cooling, the longitudinal dynamics and the cooling decrements should eventually be included in the model. However, in the present study, we focus only on the transverse dynamics described by Eqs. (53) and (54) to illustrate the calculation of the Twiss parameters and beam matrix in the context of the generalized Courant-Snyder theory. We can express Eqs. (53) and (54) in the general form of the Hamiltonian for the coupled transverse dynamics given in Eq. (1) by defining

$$\kappa \equiv \begin{pmatrix} \Omega^2 + \kappa_q + \frac{1}{\rho_x^2} & 0 \\ 0 & \Omega^2 - \kappa_q \end{pmatrix}. \tag{55}$$

The condition for stability of the particle orbits in the transverse directions can be expressed as

$$-\kappa_q \left(\kappa_q + \frac{1}{\rho_x^2}\right) > 0. \tag{56}$$

For the special case where $\kappa_q + 1/\rho_x^2 = -\kappa_q$, the transverse motions can be decoupled in the Larmor frame, and the characteristics of the particle motion are well established, e.g., in Ref. [22]. On the other hand, when $\kappa_q + 1/\rho_x^2 \neq -\kappa_q$, the transverse motions are strongly coupled in the Larmor frame, and the focusing matrix $\tilde{\kappa}$ in the Larmor frame becomes

$$\begin{aligned} \tilde{\kappa} &= Q_4 \kappa Q_4^{-1} \\ &= \begin{pmatrix} \left(\Omega^2 + \frac{1}{2\rho_x^2}\right) + \left(\kappa_q + \frac{1}{2\rho_x^2}\right)\cos 2\Omega z & \left(\kappa_q + \frac{1}{2\rho_x^2}\right)\sin 2\Omega z \\ \left(\kappa_q + \frac{1}{2\rho_x^2}\right)\sin 2\Omega z & \left(\Omega^2 + \frac{1}{2\rho_x^2}\right) - \left(\kappa_q + \frac{1}{2\rho_x^2}\right)\cos 2\Omega z \end{pmatrix}, \end{aligned} \tag{57}$$

where we have chosen the initial phase of Larmor rotation to be $\theta(z = 0) = 0$ without loss of generality. Note that the focusing matrix $\tilde{\kappa}$ is periodic with axial periodicity length $L = \pi/\Omega$, not λ of the helical magnetic field.

To demonstrate the calculation of the β function and the beam matrix for a matched beam in a helical channel, we solve the envelope matrix equation (15) numerically using the standard shooting method with Newton iteration [23]. Lengths are normalized to the lattice period λ, and we take $\Omega = 0.741$, $\rho_x = 0.147$, and $\kappa_q = -28.4$ for this example. Since the focusing matrix $\tilde{\kappa}$ has lattice period $L = 4.24$, we apply periodic boundary conditions at $z = 0$ and $z = 4.24$. Shown in Fig. 1 are the matched solutions for the elements of the beta-function matrix, $\beta = w^T w$, and the corresponding beam matrix components in the (x, y)-plane of the helical transport channel. It is interesting to note that when the beam is periodically matched, then the projections of the beam distributions onto the plane normal to the helically-rotating equilibrium orbit remain unchanged, independent of the axial position z. When $\kappa_q + 1/\rho_x^2 < -\kappa_q$, the ellipse is elongated in the x-direction, while the ellipse is elongated in the y-direction for $\kappa_q + 1/\rho_x^2 > -\kappa_q$. In Fig. 2, we plot the evolution of the matched beam over 50 helical lattice periods using the initial conditions in Fig. 1(a). It is evident that the numerically determined matching conditions are accurate enough to generate matched solutions all along the helical channel. When we introduce a mismatch by applying slight ($\lesssim 10\%$) modifications in the initial values, we observe complex envelope oscillations around the matched beam envelopes [Fig. 2(b)]. These mismatch oscillations may look benign at first glance, however, they can eventually result in emittance growth due to nonlinearities or space-charge effects present in the actual transport system [16]. Finally, we note that the tilt angle ξ of the ellipse calculated from Eq. (42) vanishes for the matched case, despite the strong coupling, while ξ becomes finite for the mismatched case.

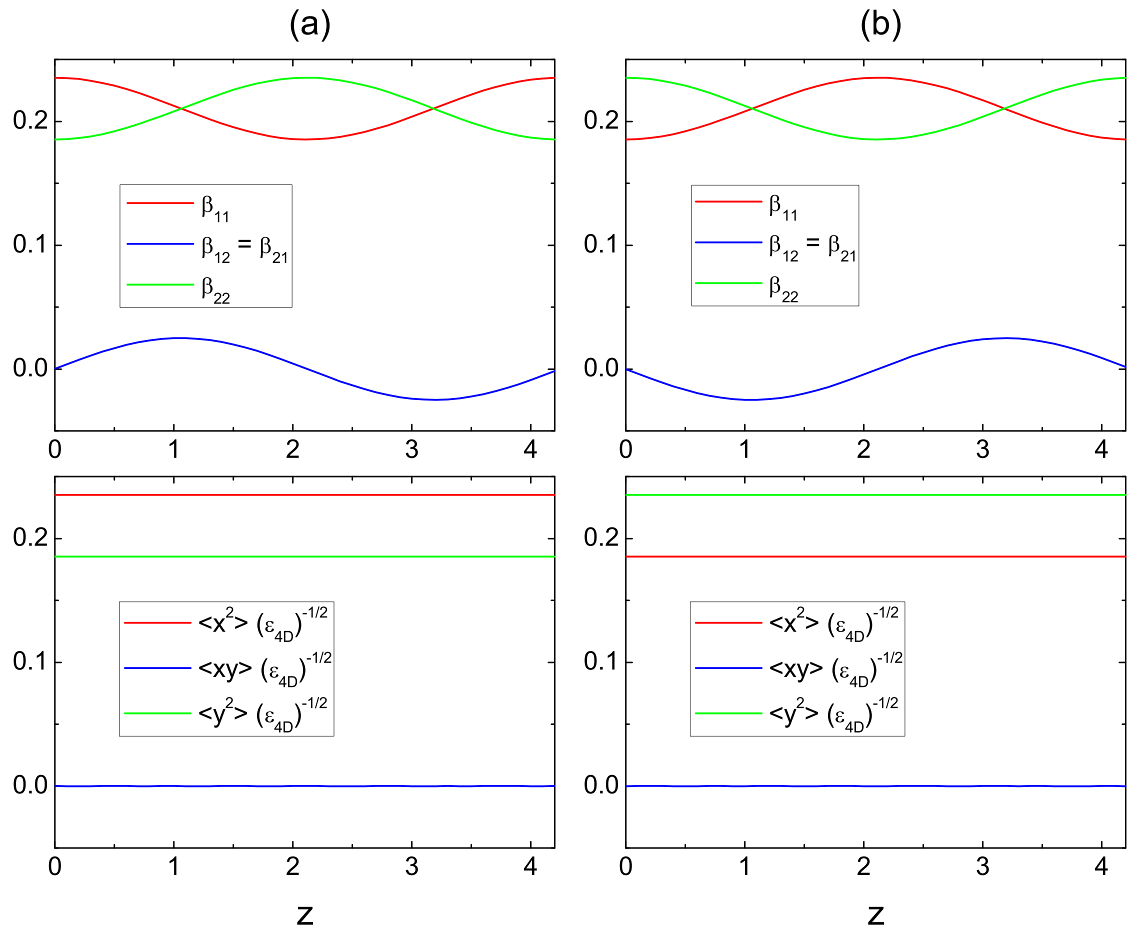

FIG. 1: The matched solutions for the elements of the beta-function matrix $\beta = w^T w$, and the corresponding beam matrix components in the (x, y)-plane of the helical transport channel for the cases with (a) $\kappa_q + 1/\rho_x^2 < -\kappa_q$, and (b) $\kappa_q + 1/\rho_x^2 > -\kappa_q$. To calculate case (b), we interchange the values of the focusing coefficients κ_x and κ_y in Eq. (55).

VII. CONCLUSIONS

Extending the generalized Courant-Snyder theory [1, 2], we have constructed the Twiss parameters and beam matrix in generalized form, which enables one to study coupled transverse dynamics within a framework similar to the original Courant-Snyder formalism. It was demonstrated that the initial conditions for the matrix envelope equation need to satisfy $(w'w^T - ww^{T'})_0 = 0$, which also simplifies the calculation of the beam matrix. In solving the

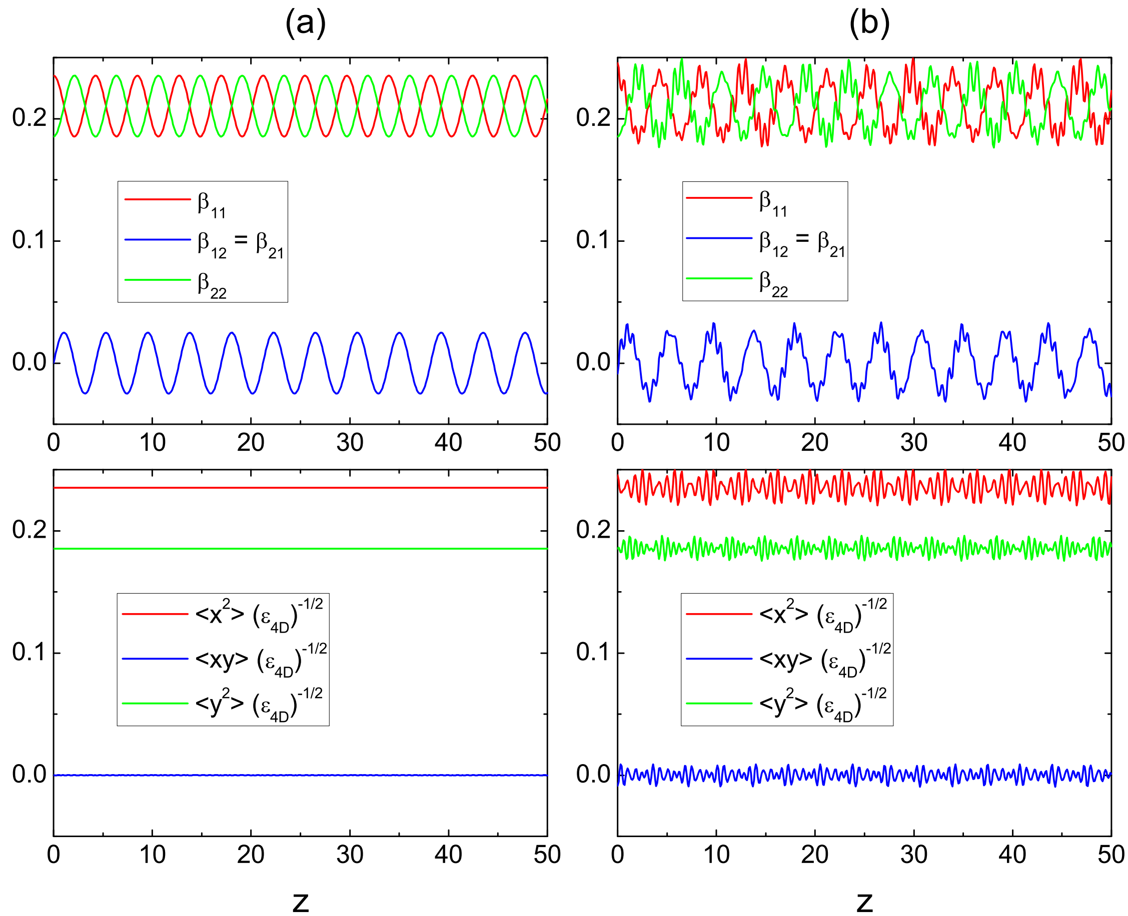

FIG. 2: Evolution of the elements of the beta-function matrix $\beta = w^T w$, and the corresponding beam matrix components in the (x, y)-plane of the helical transport channel for (a) matched, and (b) mismatched cases. For case (a), we applied the initial conditions in Fig. 1(a), while for case (b), we made slight ($\lesssim 10\%$) modifications in the initial values.

matrix envelope equation, we found that matched solutions for the envelope matrix w form an orthogonal group. However, they give a unique matched solution for the beta-function matrix $\beta = w^T w$. Furthermore, the final expressions for the Twiss parameters (α, β, and γ) and beam matrix (Σ) are remarkably similar to those of the original Courant-Snyder theory, and provide a practical framework for accelerator design, transverse beam measurement and control, and particle simulation studies.

In this study, we have focused mainly on the transverse dynamics with two degrees of freedom using 2×2 matrixes, such as w, β, and κ. However, most matrix equations developed in this paper are applicable to general $n\times n$ matrixes. Therefore, by choosing an appropriate canonical variable set u, and by constructing the focusing matrix κ accordingly, we expect to be able to extend the generalized Courant-Snyder theory to the case of three-dimensional linear coupled dynamics as well. This extension will be particularly useful for the case where both the transverse and longitudinal motions can be strongly coupled, for example, in the six-dimensional phase-space cooling experiments [20, 24] noted in Sec. VI.

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