MC-552

Longitudinal Motion with Ionization Cooling

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Abstract
Some calculations on longitudinal motion are presented. rf bucket parameters and areas are discussed, for use in developing longitudinal matching conditions and constraints for ionization cooling cases.

Longitudinal motion requires particular attention, since ionization cooling does not naturally cool effectively in the longitudinal direction, and energy straggling in the energy loss process naturally increases the energy spread to relatively large values. Also, cooling at lower energies (p_μ < 300 MeV/c) heats the beam longitudinally. The transport must keep the beam confined longitudinally, and additional longitudinal cooling (from emittance exchanges) must be regularly included in the cooling channel.

The equations of longitudinal motion in a cooling linac are:

$$\frac{d\Delta E}{ds} = eV'(\sin(\phi+\phi_s) - \sin\phi_s) \cong eV'\cos\phi_s\,\phi \tag{1}$$

$$\frac{d\phi}{ds} = \left(\frac{1}{\beta} - \frac{1}{\beta_0}\right)\frac{2\pi}{\lambda_0} \cong -\frac{1}{\beta^3\gamma^3}\frac{2\pi}{\lambda_0}\frac{\Delta E}{mc^2} \tag{2}$$

This second equation should be modified in the case of a noncollinear transport by the inclusion of the nonisochronous transport element $M_{56}' \equiv 1/\gamma_t^2$ (or $\alpha_p = 1/\gamma_t^2 - 1/\gamma^2$), to

$$\frac{d\phi}{ds} \cong \frac{1}{\beta^3\gamma}\left(\frac{1}{\gamma_t^2} - \frac{1}{\gamma^2}\right)\frac{2\pi}{\lambda_0}\frac{\Delta E}{mc^2} = C_l\Delta E\,, \tag{3}$$

where $M_{56}' = \eta/R$ and η is the dispersion and R is the local bending radius, and the symbol C_l is introduced as a shortened notation for the equation coefficient.

For the case of a collinear transport equation 2 can be rewritten as:

$$\frac{d\phi}{ds} = -\left(\frac{1}{\beta_0} - \frac{1}{\beta}\right)\frac{2\pi}{\lambda_0} = -\frac{2\pi}{\lambda_0}\left(\frac{1}{\beta_0} - \frac{\gamma}{\sqrt{\gamma^2-1}}\right) \tag{4}$$

And the first equation can be written as:

$$\frac{d\gamma}{ds} = \frac{eV'}{mc^2}(\sin(\phi+\phi_s) - \sin\phi_s) \tag{5}$$

For the case where there is no acceleration of the reference particle ($\phi_s = 0$), equations 4 and 5 can be integrated to obtain:

$$\frac{\gamma-\gamma_1}{\beta_0}+\left(\sqrt{\gamma_1^2-1}-\sqrt{\gamma^2-1}\right)=\frac{\lambda_0 eV'}{2\pi m_0 c^2}\left(\cos\phi\ -\cos\phi_1\right)$$

In ionization cooling an energy loss term dE/ds is added. The mean energy (E_0) remains constant if dE/ds = eV′ cos ϕ_s. If only this mean energy loss is included, then the equations of motion are integrable for ΔE, φ, and particle trajectories move along orbits such that:

$$\frac{(\Delta \mathrm{E})^2}{2}=\frac{\mathrm{eV}'\lambda_0 \mathrm{mc}^2\beta^3\gamma}{2\pi\alpha_\mathrm{p}}\left[(\phi-\phi_0)\sin\phi_\mathrm{s}+(\cos(\phi+\phi_\mathrm{s})-\cos(\phi_0+\phi_\mathrm{s}))\right]\ , \tag{4}$$

where ϕ_o is a constant of a trajectory and ΔE(φ) maps out a particle trajectory. The separatrix (separating trapped from untrapped particles) is obtained from eq. 4 with ϕ_0 = π- 2ϕ_s. Figure 1b shows the separatrix and an interior orbit (ϕ_0 = π-4ϕ_s) at "typical" ionization cooling parameters, used in ASOL cooling segments: eV′ = 10 MV/m, ϕ_s = 30°, βγ = 2, P(total) = 211 MeV/c, $\alpha_p = 1/\gamma^2$, and λ_o = 1.5m (200 MHz rf). The example requires inclusion of absorbers with a mean energy loss of 5 MV/m. The separatrix for this case extends over ΔE = ± 52.6 MeV.(see fig. 1) The numbers indicate that there are not large safety factors in forming stable bunch configurations at ionization cooling parameters. The maximum energy offset of the separatrix is obtained by evaluating eq.4 with ϕ_0 = π- 2ϕ_s and φ=0:

$$\Delta \mathrm{E}=\pm\sqrt{\frac{\mathrm{eV}'\lambda_0\mathrm{mc}^2\beta^3\gamma}{\pi\alpha_\mathrm{p}}\left[(2\phi_\mathrm{s}-\pi)\sin\phi_\mathrm{s}+2\cos(\phi_\mathrm{s})\right]}$$

A useful parameter is the ratio of the energy to the phase amplitude (in the small amplitude limit); this corresponds to a longitudinal "betatron" function:

$$\mathrm{R}=\frac{\phi}{\Delta\mathrm{E}}=\sqrt{\frac{2\pi\alpha_\mathrm{p}}{\mathrm{eV}'\cos\phi_\mathrm{s}\beta^3\gamma\ \lambda_0\mathrm{mc}^2}} \tag{5}$$

This factor is ~0.031 MeV^{-1} at the reference parameters. This can be changed into a distance unit by multiplying by the wavelength (λ/2π):

$$\mathrm{R}_\mathrm{s}=\frac{\delta(\mathrm{c}\tau)}{\Delta\mathrm{E}}=\sqrt{\frac{\alpha_\mathrm{p}\lambda_0}{2\pi\mathrm{eV}'\cos\phi_\mathrm{s}\beta^3\gamma\mathrm{mc}^2}} \tag{6}$$

This becomes more properly a longitudinal betatron function when the variable δp/p = ΔE/($\beta^2\gamma$mc^2) is used:

$$\mathrm{R}_{\mathrm{c}\tau-\delta\mathrm{p/p}}=\frac{\delta(\mathrm{c}\tau)}{\delta\mathrm{p}/\mathrm{p}}=\sqrt{\frac{\alpha_\mathrm{p}\lambda_0\beta\gamma\mathrm{mc}^2}{2\pi\mathrm{eV}'\cos\phi_\mathrm{s}}}$$

Another useful parameter is the longitudinal motion oscillation length-the distance over which the longitudinal motion undergoes a full phase-space oscillation. For small amplitude oscillations this can be written as:

$$\lambda_{\mathrm{osc}} = 2\pi\sqrt{\frac{\beta^3\gamma\lambda_0 \mathrm{mc}^2}{2\pi\alpha_{\mathrm{p}}\mathrm{eV}'\cos\phi_{\mathrm{s}}}} \, . \tag{7}$$

This is ~40m at the reference parameters.

As previously discussed, the rms energy spread is naturally at least 3% from energy straggling with cooling. Thus an rms energy spread of ΔE_{rms} ~12MeV (3σ = 36 MeV), would fit within the rf bucket shown in Fig. 1 and would have a 3σ phase spread of ±1.1 radian ($\phi_{rms} \cong 0.37$ radians or $\delta ct \cong 8.8$cm, which would lead to a normalized longitudinal emittance of $\varepsilon_{L,rms} \cong \delta ct\, \Delta E_{rms}/(mc^2) \cong 0.01$m. These parameters may be considered in an initial cooling scenario. The beam should be initially trapped within the bucket and the beam would remain within the bucket for a significant distance before energy straggling causes beam loss.

In the Study 2 scenario the beam rms energy spread is 24 MeV, which places the rf bucket border at 2σ. A bunch length twice as long (18cm rms) would be correspondingly matched. In this case, as in Study I, the beam would initially fill the cooling bucket and leak from it with straggling as the beam propagates down the cooling channel.

Longitudinal motion in a helical cooling channel follows similar equations. The equations of motion are:

$$\frac{d\Delta E}{dz} = eV'(\sin(\phi+\phi_s) - \sin\phi_s) \cong eV'\cos\phi_s\,\phi$$

$$\frac{d\phi}{dz} = -\eta_H \frac{2\pi}{\lambda_0}\frac{\Delta E}{mc^2}$$

where η_H is given by: $\eta_H = \frac{\sqrt{1+\kappa^2}}{\gamma\beta^3}\left[\frac{\hat{D}\kappa^2}{1+\kappa^2} - \frac{1}{\gamma^2}\right]$.

and κ is the helical pitch and $\hat{D}$ is the helical dispersion factor. z, the longitudinal cooridinate along the helix axis, is used as the independent variable. The same equations for phase space follow, with $\alpha_p/(\beta^3\gamma)$ replaced by η_H.

The stable phase space area can be found by integrating $2\Delta E$ from minimum to maximum φ (using eq. 4). For the case of $\phi_s = 0$, This area (in ΔE-.ϕ) is:

$$A = 8\sqrt{\frac{2eV'\lambda_0 mc^2\beta^3\gamma}{\pi\alpha_p}}$$

We can change the unit to meters by multiplying by $\lambda_0/2\pi$ and dividing by mc^2, obtaining:

$$A = \frac{4\lambda_0}{\pi}\sqrt{\frac{2eV'\lambda_0\beta^3\gamma}{mc^2\pi\alpha_p}}$$

For the baseline numbers (V' = 10 MV/m, $\beta\gamma$=2, λ_0=1.5m mc^2=105.66 MeV) this is 1.624m. To change this full beam size to an rms unit (σ_z σ_E/mc^2) we might divide by 4π, obtaining 0.13m. If φ_s is nonzero, one can integrate 2ΔE numerically, obtaining a smaller area than at ϕ_s = 0. That reduction factor as a function of φ_s is shown in figure 2. The stable phase area decreases rapidly with increasing φ_s At $\varphi_s = \pi/6$ or 30°, a commonly used stability point, the phase area is ~0.33 of that at φ_s = 0 (0.043m in rms estimates at baseline parameters.).

This reduction is an important part of the difficulty in transition from stationary to accelerating (or cooling) beam dynamics, and is a major cause of losses in these transition. Even small changes reduce the stable phase area significantly. (The area is reduced by ~33% at φ_s =10°.) S. Y. Lee has approximated this area change factor as:

$$\frac{1+\sin(\phi_s)}{1-\sin(\phi_s)},$$

which is a fairly close approximation to the numerical integration for φ_s<1.5.

Matching of longitudinal betatron functions is obtained simply when V' cos φ_s is matched; but stable phase space area is not then matched when φ_s changes..

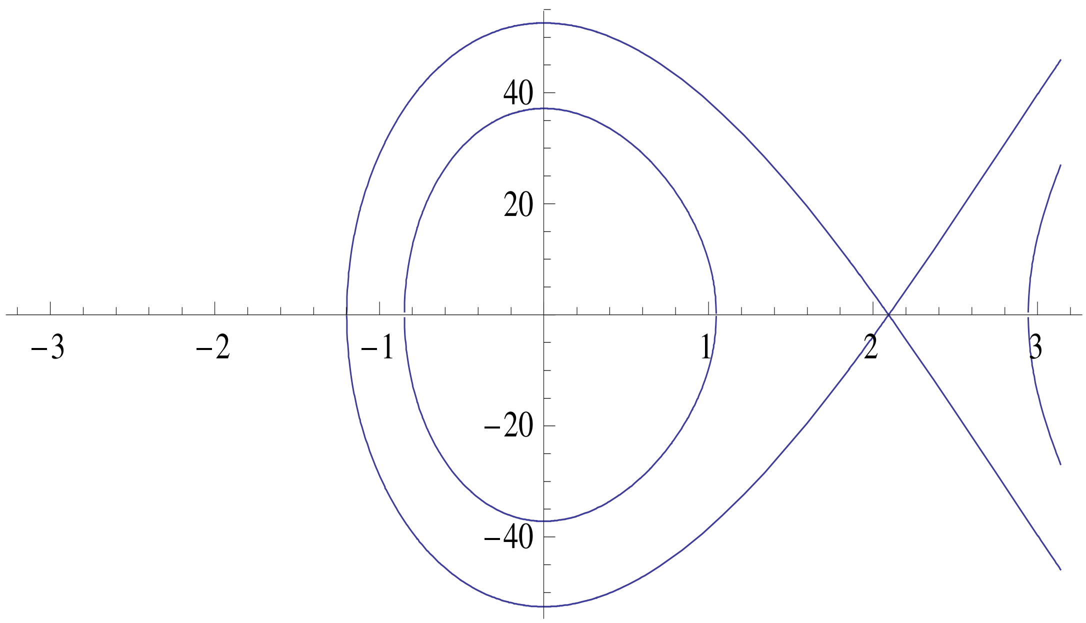
The equations discussed above can be used to develop other parameters for longitudinal beam matching.

References

1. S. Y. Lee, Accelerator Physics, p. 229, World Scientific (1999).

2. Y. Derbenev and R. Johnson, Phys. Rev. STAB 8 041002 (2005).

Figure 1: rf bucket shape for P = 212 MeV/c muons at parameters of eV′ = 10 MV/m, ϕ_s =30°, E α_p =1/γ^2, and λ_o = 1.5m (200 MHz rf). Phase space area of the rf bucket is 237MeV-radians or 0.54m.



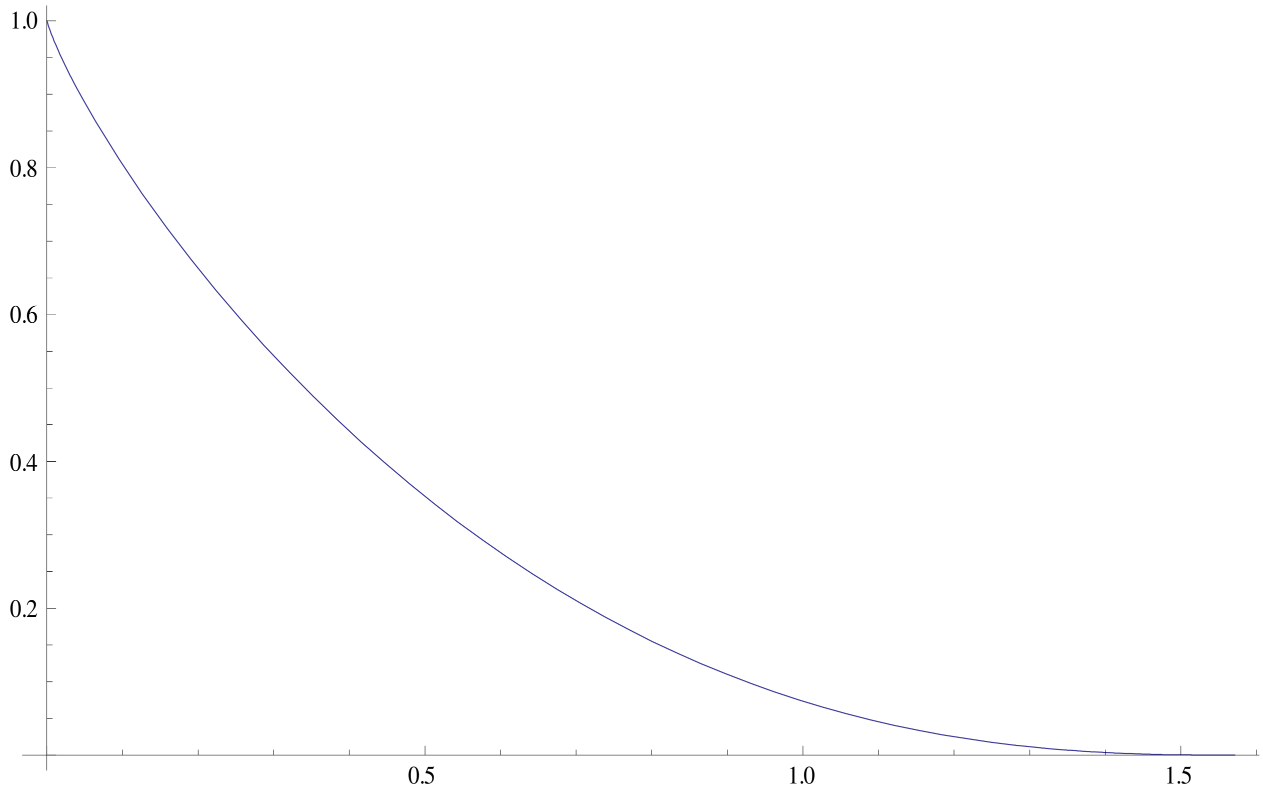


Figure 2. Reduction of the stable phase space area (in φ-δE) of the rf bucket as a function of stable phase φ_s in radians. Operation at $\varphi_s = 30°$ reduces the rf bucket area to ~1/3 of the φ_s=0 value.

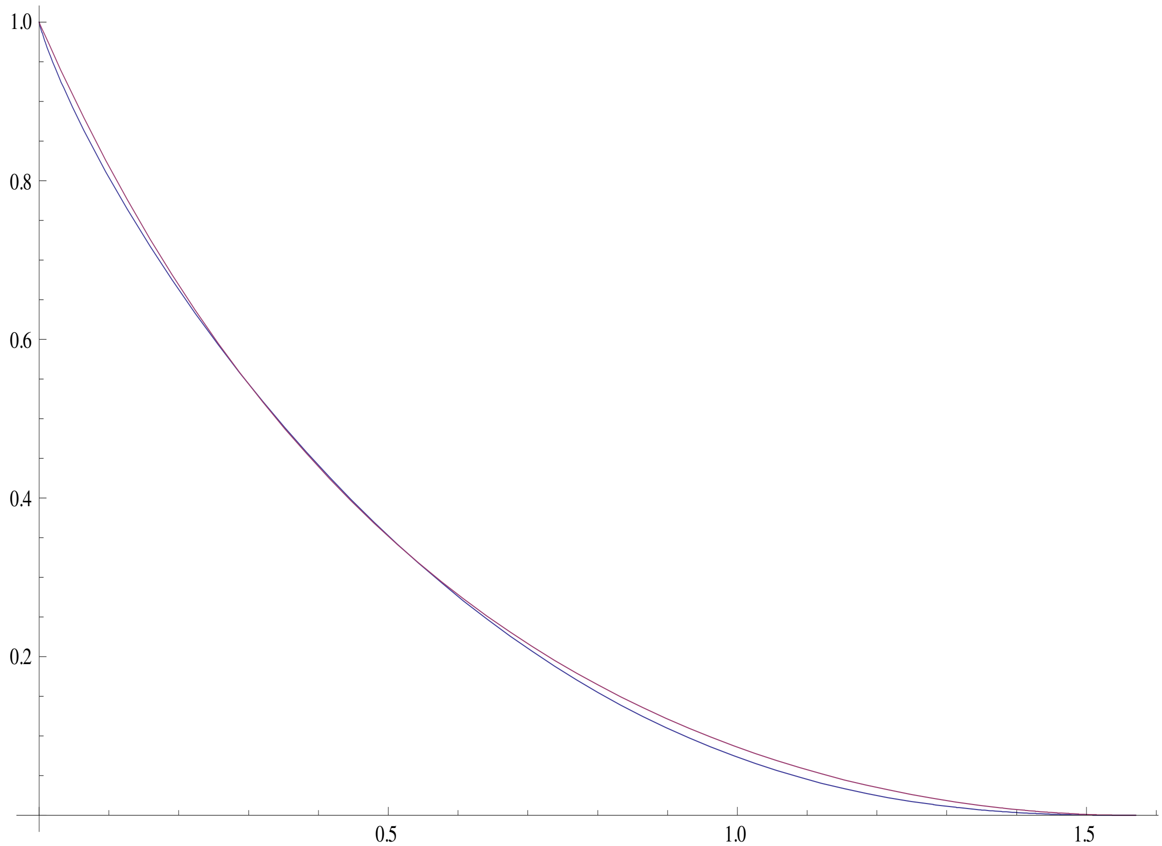


Figure 3: Same plot but with $(1-\sin(\varphi_s))/(1+\sin(\varphi_s))$ also shown in red