Linear model for non-isosceles absorbers

J. Scott Berg*
Brookhaven National Laboratory; Building 901A; PO Box 5000; Upton, NY 11973-5000
(Dated: 30 October 2002)

Previous analyses have assumed that wedge absorbers are triangularly shaped with equal angles for the two faces. In this case, to linear order, the energy loss depends only on the position in the direction of the face tilt, and is independent of the incoming angle. One can instead construct an absorber with entrance and exit faces facing rather general directions. In this case, the energy loss can depend on both the position and the angle of the particle in question. This paper demonstrates that and computes the effect to linear order.

I. INTRODUCTION

Ionization cooling can be achieved in the transverse direction with a straight cooling channel. However, in the longitudinal direction, one at best gets very slow cooling, and in most cases actually gets heating. In addition, energy straggling leads to further heating in the longitudinal plane. To achieve 6-D cooling, one must couple the transverse motion with longitudinal motion. One method to achieve this is to use a triangular cross-section absorber in a location with dispersion. Particles with higher energy then go through a larger length of absorber and lose more energy, thus reducing the energy spread. Unfortunately, this occurs at the cost of an increase in transverse beam size [1]. This process is often referred to as “emittance exchange.”

Existing computations have only considered triangular wedges with equal face tilts. The entrance and exit faces of the absorber can be tilted rather generally. This will give an energy loss dependence on transverse coordinates which is different from what occurs when the face tilts are equal and in the same plane. This paper calculates the linear transfer matrix for such a wedge absorber, for use in theoretical calculations.

First, the path length in the absorber is calculated for general face angles. The computation is first done in the case where the faces are tilted in the same plane, to give a more intuitive picture of what is going on, followed by formulas for more general face tilts. This calculation is then used to find the transfer matrix through the absorber. Finally, possible uses of more general face angles are discussed, in particular the case where the faces are parallel but tilted.

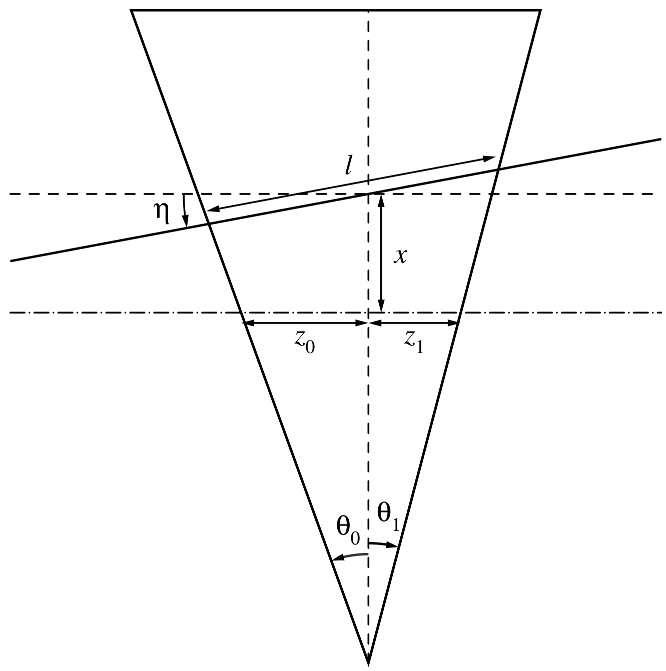

FIG. 1: Planar absorber geometry, showing relevant parameters. x is the displacement of the orbit from the reference axis at the center of the absorber, and η is the angle that the incoming particle makes with the reference axis.

II. GEOMETRIC LENGTH CALCULATION

The energy loss (ignoring stochastic effects) in the absorber is proportional to the distance that the particle travels through the absorber. Thus, to calculate the effect of the absorber, we will simply calculate the length of the particle trajectory that is inside the absorber. We further assume that the particle trajectory is straight: i.e., there are no electromagnetic fields acting on the particle. First, we compute a case where the plane in which the absorber faces are tilted from vertical is the same for both cases. This becomes a one-dimensional problem, and helps give an understanding of what is going on. We then do the computation for a more general case.

A. One Dimension

For one dimension we can do the calculation in a plane. Fig. 1 is a diagram giving the relevant parameters. The exact expression for the length of the path inside the

*Electronic address: `jsberg@bnl.gov`; URL: `http://pubweb.bnl.gov/people/jsberg/`

absorber is

$$l = \left(\frac{L}{\tan\theta_0 + \tan\theta_1} + x\right)\frac{\cos\eta\sin(\theta_0+\theta_1)}{\cos(\eta-\theta_0)\cos(\eta+\theta_1)}, \quad (1)$$

where $L = z_1 + z_2$. To linear order in x and η, this is

$$l = L + x(\tan\theta_0 + \tan\theta_1) + \eta L(\tan\theta_1 - \tan\theta_0) \quad (2)$$

If $\theta_0 = \theta_1$ (i.e., the absorber cross-section is an isosceles triangle), to linear order, the path length (and thus the energy loss) does not depend on the incoming particle angle, but does depend on the transverse position. This is the situation that has been analyzed in the past. On the other hand, if $\theta_0 = -\theta_1$ (i.e., the faces are parallel but the absorber is tilted), the path length does not depend on the incoming particle position, but it does depend on the incoming particle angle. Note that Eq. (1) cannot be evaluated at $\theta_0 = -\theta_1 = \theta$; one must take the limit, which exists and is

$$L\frac{\cos\eta\cos^2\theta}{\cos^2(\eta-\theta)} \quad (3)$$

B. General Geometry

In the more general situation, we describe the absorber by its entrance and exit planes. We describe these planes as passing through a point p_i and having a unit normal u_i, where $i = 0$ for the entrance plane and $i = 1$ for the exit plane. The particle trajectory is described as line passing through a point x_0 with a unit tangent vector t. We can calculate the path length within the absorber in terms of these coordinates and vectors as

$$l = \frac{(\boldsymbol{p}_1 - \boldsymbol{x}_0)\cdot\boldsymbol{u}_1}{\boldsymbol{t}\cdot\boldsymbol{u}_1} - \frac{(\boldsymbol{p}_0 - \boldsymbol{x}_0)\cdot\boldsymbol{u}_0}{\boldsymbol{t}\cdot\boldsymbol{u}_0} \quad (4)$$

We can compute this length to linear order. It is useful at this point to represent the aforementioned vectors and points in terms of coordinates:

$$\boldsymbol{x}_0 = (x, y, 0) \quad (5)$$

$$\boldsymbol{t} = \left(\frac{p_x}{p}, \frac{p_y}{p}, \frac{\sqrt{p^2 - p_x^2 - p_y^2}}{p}\right) \quad (6)$$

$$\boldsymbol{p}_0 = (0, 0, -z_0) \quad (7)$$

$$\boldsymbol{p}_1 = (0, 0, z_1) \quad (8)$$

$$\boldsymbol{u}_0 = (\sin\theta_0\cos\phi_0, \sin\theta_0\sin\phi_0, \cos\theta_0) \quad (9)$$

$$\boldsymbol{u}_1 = (-\sin\theta_1\cos\phi_1, -\sin\theta_1\sin\phi_1, \cos\theta_1). \quad (10)$$

These values have been chosen to match the values in Fig. 1. In terms of these, the path length to linear order in the transverse variables is

$$\begin{aligned} l = z_0 + z_1 &+ x(\tan\theta_0\cos\phi_0 + \tan\theta_1\cos\phi_1) \\ &+ y(\tan\theta_0\sin\phi_0 + \tan\theta_1\sin\phi_1) \\ &+ \frac{p_x}{p}(z_1\tan\theta_1\cos\phi_1 - z_0\tan\theta_0\cos\phi_0) \\ &+ \frac{p_y}{p}(z_1\tan\theta_1\sin\phi_1 - z_0\tan\theta_0\sin\phi_0). \end{aligned} \quad (11)$$

Note that in contrast with Eq. (3), this equation does not depend only in $L = z_0 + z_1$. This is because for the planar case, it was possible to choose a single longitudinal position corresponding to the point at which the entrance and exit planes meet (therefore specifying all of θ_0, θ_1, z_0, and z_1 is redundant). That is not possible in general, and thus we must keep z_1 and z_2 separate.

III. TRANSFER MATRIX

One can easily compute the transfer matrix to lowest order in the relative energy loss in the absorber. In this case, only the path length in the absorber matters. First, compute the evolution of the transverse momenta, as well as the evolution of the energy deviation ignoring the face angles. The equations of motion are

$$\frac{d\boldsymbol{p}_\perp}{ds} = -\kappa_\perp\boldsymbol{p}_\perp \qquad \frac{d\delta}{ds} = -\kappa_\parallel\delta \quad (12)$$

$$\kappa_\perp = \frac{1}{\beta pc}\frac{dE}{dx} \qquad \kappa_\parallel = \frac{d}{dE}\left(\frac{dE}{dx}\right), \quad (13)$$

and their solution is

$$\boldsymbol{p}_\perp(s) = \boldsymbol{p}_\perp(s_0)e^{-\kappa_\perp(s-s_0)} \quad (14)$$

$$\delta(s) = \delta(s_0)e^{-\kappa_\parallel(s-s_0)}. \quad (15)$$

The change in δ due to the pole faces can be computed from Eq. (11) to be

$$\begin{aligned} \Delta\delta = \kappa_\perp\Big[&x(\tan\theta_0\cos\phi_0 + \tan\theta_1\cos\phi_1) \\ &+ y(\tan\theta_0\sin\phi_0 + \tan\theta_1\sin\phi_1) \\ &+ \frac{p_x}{p}(z_1\tan\theta_1\cos\phi_1 - z_0\tan\theta_0\cos\phi_0) \\ &+ \frac{p_y}{p}(z_1\tan\theta_1\sin\phi_1 - z_0\tan\theta_0\sin\phi_0)\Big]. \end{aligned} \quad (16)$$

The matrix elements can be read off from the above equations. In a theoretical treatment, this matrix element can be treated as having zero length, thus ignoring the drift. This is because the particle trajectory angles do not change due to the energy loss. Also remember that the effects of fields in the absorbers are ignored.

This calculation ignores the evolution of the transverse momenta as given in Eq. (14), which will result in a small relative error of order $\kappa_\perp L$. One can do the correct calculation, at least in one dimension. Take z to be the position along the reference axis and θ the angle of the plane with respect to vertical. There is a relationship between these: $z = y\tan\theta$ where y is a constant. Then

$$dl = \left(1 + \frac{x}{y} + 2\frac{p_x z}{py}\right) dz \tag{17}$$

is the infinitesimal length change. One can use Eq. (14) in this equation, giving

$$\frac{dl}{dz} = 1 + \frac{x}{y} + 2\frac{p_{x0}}{py} z e^{-\kappa_\perp(z+z_0)}. \tag{18}$$

Integrating this, one gets

$$l = L + \frac{xL}{y} + 2\frac{p_{x0}}{py}\left[\frac{1}{\kappa_\perp^2}(1 - e^{-\kappa_\perp L}) - \frac{z_1}{\kappa_\perp}e^{-\kappa_\perp L} - \frac{z_0}{\kappa_\perp}\right] \tag{19}$$

This can be rewritten in terms of angles and L as

$$l = L + x(\tan\theta_0 + \tan\theta_1) + 2\frac{p_x}{p\kappa_\perp}\left[(\tan\theta_0 + \tan\theta_1)\frac{1 - e^{-\kappa_\perp L}}{\kappa_\perp L} - \tan\theta_1 e^{-\kappa_\perp L} - \tan\theta_0\right] \tag{20}$$

Doing this in the general case is more difficult, but results in an equally simple answer. We need to imagine a sequence of planes with normal $\boldsymbol{u}$ and passing through the point $(0, 0, z)$. To linear order in $\boldsymbol{x} = (x, y)$ and $\boldsymbol{p}_\perp/p$ (where the $\perp$ subscript refers to the transverse coordinates), the integrated path length to the plane at z is a constant plus

$$z - \boldsymbol{x}\cdot\frac{\boldsymbol{u}_\perp}{u_z} - z\frac{\boldsymbol{p}_\perp}{p}\cdot\frac{\boldsymbol{u}_\perp}{u_z}. \tag{21}$$

We will parameterize $\boldsymbol{u}$ by η according to

$$\boldsymbol{u} = \frac{\boldsymbol{u}_0\sin(\xi-\eta) + \boldsymbol{u}_1\sin\eta}{\sin\eta} \qquad \boldsymbol{u}_0\cdot\boldsymbol{u}_1 = \cos\xi. \tag{22}$$

η will vary from 0 to ξ while z varies from $-z_0$ to z_1. The definition of the exact relationship between them will be left to later. We then have

$$\frac{dl}{dz} = 1 - \frac{\boldsymbol{p}_\perp}{p}\cdot\frac{\boldsymbol{u}_\perp}{u_z} - \left(\boldsymbol{x} + z\frac{\boldsymbol{p}_\perp}{p}\right)\cdot\frac{u_{z0}\boldsymbol{u}_{\perp 1} - u_{z1}\boldsymbol{u}_{\perp 0}}{u_z^2\sin\xi}\frac{d\eta}{dz}. \tag{23}$$

Now define

$$\frac{d\eta}{dz} = k u_z^2 \tag{24}$$

for some constant k. Then

$$\frac{d}{dz}\left(\frac{\boldsymbol{u}_\perp}{u_z}\right) = k\frac{u_{z0}\boldsymbol{u}_{\perp 1} - u_{z1}\boldsymbol{u}_{\perp 0}}{\sin\xi} \tag{25}$$

meaning that

$$\frac{\boldsymbol{u}_\perp}{u_z} = k\frac{u_{z0}\boldsymbol{u}_{\perp 1} - u_{z1}\boldsymbol{u}_{\perp 0}}{\sin\xi} + \boldsymbol{c} \tag{26}$$

for a constant vector $\boldsymbol{c}$. Applying the known boundary conditions,

$$\frac{\boldsymbol{u}_\perp}{u_z} = \frac{\boldsymbol{u}_{\perp 0}}{u_{z0}}\frac{z_1 - z}{L} + \frac{\boldsymbol{u}_{\perp 1}}{u_{z1}}\frac{z + z_0}{L}. \tag{27}$$

We can then write

$$\frac{dl}{dz} = 1 - \frac{\boldsymbol{x}}{L}\cdot\left(\frac{\boldsymbol{u}_{\perp 1}}{u_{z1}} - \frac{\boldsymbol{u}_{\perp 0}}{u_{z0}}\right) - \frac{\boldsymbol{p}_\perp}{p}\cdot\left[\frac{z_1\boldsymbol{u}_{\perp 0}}{Lu_{z0}} + \frac{z_0\boldsymbol{u}_{\perp 1}}{Lu_{z1}} + \frac{2z}{L}\left(\frac{\boldsymbol{u}_{\perp 1}}{u_{z1}} - \frac{\boldsymbol{u}_{\perp 0}}{u_{z0}}\right)\right] \tag{28}$$

Now, use Eq. (14), giving

$$l = L - \boldsymbol{x}\cdot\left(\frac{\boldsymbol{u}_{\perp 1}}{u_{z1}} - \frac{\boldsymbol{u}_{\perp 0}}{u_{z0}}\right) - \frac{\boldsymbol{p}_{\perp 0}}{p\kappa_\perp L}\cdot\Bigg[2\left(\frac{1 - e^{-\kappa_\perp L}}{\kappa_\perp} - z_0 - z_1 e^{-\kappa_\perp L}\right)\left(\frac{\boldsymbol{u}_{\perp 1}}{u_{z1}} - \frac{\boldsymbol{u}_{\perp 0}}{u_{z0}}\right) + \left(\frac{z_1\boldsymbol{u}_{\perp 0}}{u_{z0}} + \frac{z_0\boldsymbol{u}_{\perp 1}}{u_{z1}}\right)(1 - e^{-\kappa_\perp L})\Bigg]. \tag{29}$$

The change in δ is simply $\kappa_\perp l$, and one can then directly read off the matrix elements.

IV. DISCUSSION

One can now ask the question of why this is interesting. The simplest point is that adjusting the absorber geometry simply to keep the sum of the tangents of the face angles constant is not sufficient to maintain identical performance of a cooling channel, unless there happens to be no angular dispersion at the location of the absorber. In fact, at a point where there is both angular and positional dispersion, one can potentially get improved performance out of a given absorber by adjusting the face angles separately.

One could even consider a lattice with only angular dispersion and no positional dispersion at the absorbers. This could not be easily done in a ring (if one bends in the same direction all the time, one tends to have nonzero positional dispersion), but could be done in a "snaking" configuration where subsequent cells bend in opposite directions, and thus the lattice is straight over larger scales.

There are several reasons one might want to do this. First of all, a lattice that does not form a ring allows one to adiabatically vary lattice parameters, thus maximizing the cooling performance as the beam changes. One may be especially interested in doing this for a collider to maximize the luminosity one achieves. In addition, one avoids the difficulties with injection and extraction.

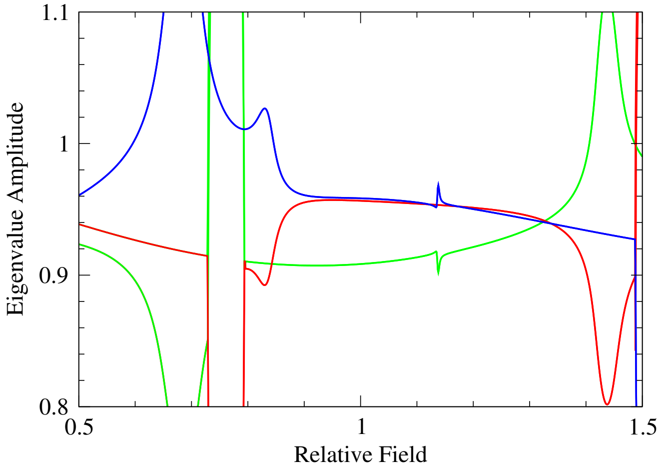

FIG. 2: Super-FOFO lattice with bends all in same direction and isosceles wedges. Absolute value of eigenvalues plotted versus field relative to a reference field. Particles on-axis lose 5% of their momentum in one absorber (without coupling, the energy spread would increase by 0.85%). Bend radius at reference field is 5 m (momentum of 200 MeV/c). Wedge faces tilted vertically by 55°. RF is 201.25 MHz, 50° off-crest, 3 MV/m.

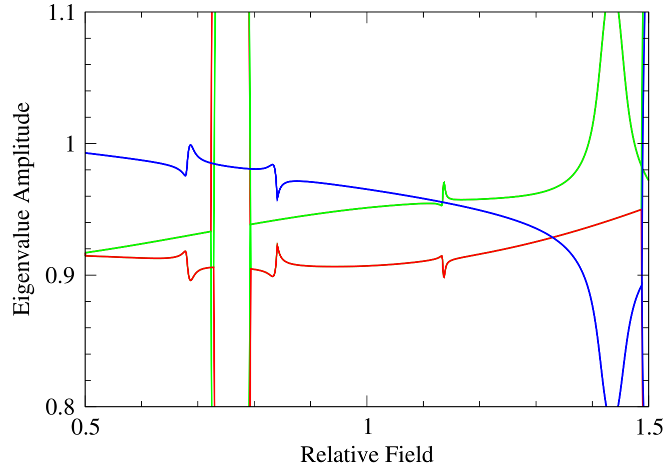

FIG. 3: Super-FOFO lattice where bending in one cell is opposite to the bending in the next, and the absorbers are slabs tilted horizontally by 75°. Reference bend radius is 3.5 m. Everything else as in Fig. 2.

These considerations apply to any lattice that does not form a ring. The advantage of having only angular dispersion versus positional dispersion at the absorber may lie in the effect of energy straggling. When the energy changes in energy straggling, the betatron amplitude will change since the closed orbit changes. Since the beta function at the absorber is small (whereas the dispersion is not necessarily), energy straggling with positional dispersion will lead to large betatron amplitude changes relative to the beam size, since the beam size is small due to the small beta function. However, if instead there is angular dispersion at the absorber, energy straggling leads to smaller relative betatron amplitude changes due to the large angular spread at that point. This has the potential to substantially improve the performance of these cooling lattices. This has not been tried out in real lattices at this point, so this discussion is speculative.

A. Example

As an example, consider a "Super-FOFO" lattice [2], modified by adding bending as in [3]. A similar lattice, but with "RFOFO" cooling cells, has been proposed for achieving 6-D cooling, and shows excellent performance [4, 5]. Fig. 2 shows the absolute value of the eigenvalues as a function of the field strengths for standard isosceles wedges and a lattice where all bends bend in the same direction (giving dispersion at the absorber). This is equivalent to considering the dependence of the eigenvalues on the reference momentum. If the absolute value of all the eigenvalue is less than 1, then the beam will be cooled in all planes. As one can see from that figure, one is able to achieve 6-D cooling over a rather large range of reference momenta. One therefore can speculate that 6-D cooling will occur over a fairly large longitudinal phase-space volume.

Figure 3 shows the eigenvalues for a tilted slab in a lattice that has angular dispersion at the absorber. Note that in this case as well, one is able to achieve 6-D cooling over a rather large range of reference momenta. Also note that the wedge angles are steeper than those required for the case with conventional wedge absorbers.

V. CONCLUSIONS

The energy loss in an absorber with generally placed planar faces has been calculated to linear order in the transverse coordinates. This allows one to calculate eigenvalues for a cooling channel with these rather general wedges. An example was constructed where a cooling channel was constructed with angular dispersion at the absorbers, and parallel-face tilted absorbers were used. Linear performance (without multiple scattering) of that cooling channel was shown to be comparable to that of a channel constructed with more conventional wedges. It can be speculated that such a channel has multiple scattering performance that is better than a wedge-based 6-D cooling channel.

Further work should incorporate general face orientations into simulation codes such as ICOOL [6]. One can then examine the cooling performance of real lattices with absorbers with more generally placed faces. One can also attempt to optimize proposed 6-D cooling lattices by orienting absorber faces more generally.

[1] David Neuffer, Part. Accel. **14**, 75 (1983).

[2] Eun-San Kim *et al.*, report MUC-NOTE-COOL_THEORY-0036, http://www-mucool.fnal.gov/notes/notes.html (unpublished).

[3] J. Scott Berg, in *Proceedings of the 2001 Particle Accelerator Conference*, edited by P. Lucas and S. Webber (IEEE, Piscataway, NJ 2001), p. 145.

[4] J.S. Berg, R.C. Fernow, R.B. Palmer, MUC-NOTE-COOL_THEORY-0239, http://www-mucool.fnal.gov/notes/notes.html (unpublished).

[5] R.B. Palmer, MUC-NOTE-COOL_THEORY-0250, http://www-mucool.fnal.gov/notes/notes.html (unpublished).

[6] R. Fernow, computer program ICOOL, http://pubweb.bnl.gov/people/fernow/icool/readme.htm (unpublished).