NOTE ON CURVED LITHIUM LENS

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Abstract

We give the analytical expression of the magnetic field generated by an idealized bent lithium lens. The self-consistent current density vector is of the form $\mathbf{J} = \frac{J_o}{2}\left\{\frac{R_o}{R} + \left(\frac{R_o}{R}\right)^2\right\}\mathbf{e}_\phi$. All quantities have the right limit when $R_o \to \infty$.

1 INTRODUCTION

A lithium lens with an specified curvature has being proposed to be used in cooling rings to achieve transverse and longitudinal emittances appropriate for a Muon Collider [1], [2] and [3].

2 STRAIGHT LENS

An idealized model of a straight lithium lens calls for a uniform current density $\mathbf{J} = (0, J_o, 0)$ in the body of the device.

Maxwell's equation $\nabla \times \mathbf{B} = \mu_o \mathbf{J}$ implies:

$$B_r = B_z = 0 \qquad \text{and} \qquad \boxed{B_\theta = \frac{\mu_o}{2\pi}\frac{I}{r} = \begin{cases} \frac{\mu_o J_o}{2} r & r < a \\ \frac{\mu_o J_o}{2}\frac{a^2}{r} & r > a \end{cases}} \tag{1}$$

where I is the total current and a is the radius of the lens.

We introduce the vector potential $\nabla \times \mathbf{A} = \mathbf{B}$ which gives

$$B_r = \frac{1}{r}\frac{\partial A_z}{\partial \phi} \qquad B_\theta = -\frac{\partial A_z}{\partial r}.$$

By observation we can write that the longitudinal component of the vector potential

$$\boxed{A_z = -\frac{\mu_o}{4} J_o r^2}. \tag{2}$$

Note that trivially, $\nabla \cdot \mathbf{A} = 0$ and $\nabla \cdot \mathbf{B} = 0$ as it should.

3 BENT LENS

For the problem at hand it is helpful to introduce a variation of a *toroidal coordinate system* as shown in Fig. 1. The relation between coordinate systems is:

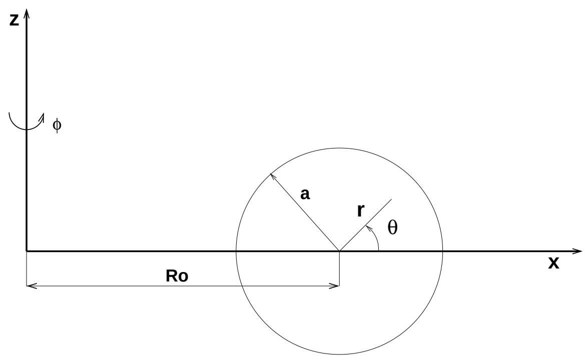

Figure 1: Definition of the toroidal coordinates (r, θ, ϕ).

$$R = R_o + r\cos\theta \tag{3}$$

$$\mathbf{r} = R\cos\phi\,\mathbf{e_x} + R\sin\phi\,\mathbf{e_y} + r\sin\theta\,\mathbf{e_z} \tag{4}$$

where R_o is the radius of curvature of the curved lens. We will need the scale factors [4] $h_1^2 \equiv 1$, $h_2^2 \equiv r^2$ and $h_3^2 = R^2$ and the expression for the curl of a vector $\mathbf{F}$

$$\nabla \times \mathbf{F} = \frac{1}{rR}\left\{\mathbf{e_r}\left[\frac{\partial(RF_\phi)}{\partial\theta} - \frac{\partial(rF_\theta)}{\partial\phi}\right]\right. \tag{5}$$

$$+r\,\mathbf{e}_\theta\left[\frac{\partial F_r}{\partial\phi} - \frac{\partial(RF_\phi)}{\partial r}\right] \tag{6}$$

$$\left.+R\,\mathbf{e}_\phi\left[\frac{\partial(rF_\theta)}{\partial r} - \frac{\partial F_r}{\partial\theta}\right]\right\} \tag{7}$$

$$\tag{8}$$

and for the divergence of a vector **F**

$$\nabla \cdot \mathbf{F} = \frac{1}{rR} \left\{ \frac{\partial (rRF_r)}{\partial r} + \frac{\partial (RF_\theta)}{\partial \theta} + \frac{\partial (rF_\phi)}{\partial \phi} \right\} \tag{9}$$

Now we guess an expression for the vector potential $A_\phi(r, \theta)$ such that in the limit $R_o \to \infty$ we recover the expression of A_z (*note the change in labels of the cartesian axis*).

It is natural to take $A_\phi(r, \theta) = -\mu_o \frac{J_o}{4} r^2 \left(\frac{R_o}{R} \right)$, then

$$\begin{aligned} B_r &= \frac{1}{rR} \frac{\partial (RA_\phi)}{\partial \theta} = 0 \\ B_\theta &= -\frac{1}{R} \frac{\partial (RA_\phi)}{\partial r} = \mu_o \frac{J_o}{2} \left(\frac{R_o}{R} \right) r \quad \text{for} \quad r < a. \end{aligned} \tag{10}$$

Trivially, $\nabla \cdot \mathbf{A} = 0$ and $\nabla \cdot \mathbf{B} = 0$.

Now we have to find the current density vector **J** in this toroidal geometry; using Maxwell's equation $\nabla \times \mathbf{A} = \mu_o \mathbf{J}$ and some algebra we find $J_\phi = \frac{J_o}{2} \left\{ \frac{R_o}{R} + \left(\frac{R_o}{R} \right)^2 \right\}$. Obviously, in the limit $R_o \to \infty$ we obtain the correct uniform current J_o.

Distintive features of the solution $\mathbf{B}_\theta$ are:

- the field is zero at the center of the lithium lens ($r = 0$)
- $B_z = -B_\theta \cos\theta$ is constant on planes of constant $x - R_o = r\cos\theta$ and the magnitude increases as we move left toward the geometric center of the toroid, reaching maximum at $x_{max} = R_o - a$ with magnitude $B_z = -\mu_o \frac{J_o}{2} \frac{R_o a}{R_o - a}$ (see Fig. 2)

We show in Fig. 3 the magnetic field for two different radius of curvature R_o of a lithium lens with radius $a = 0.1$ m.

References

[1] Y. Fukui, et.al. Overview of Recent Progress on 6D Muon Cooling with Ring Coolers, 341-v1

[2] Y. Fukui, et.al. A Muon Cooling Ring with Curved Lithium Lenses, 328-v1

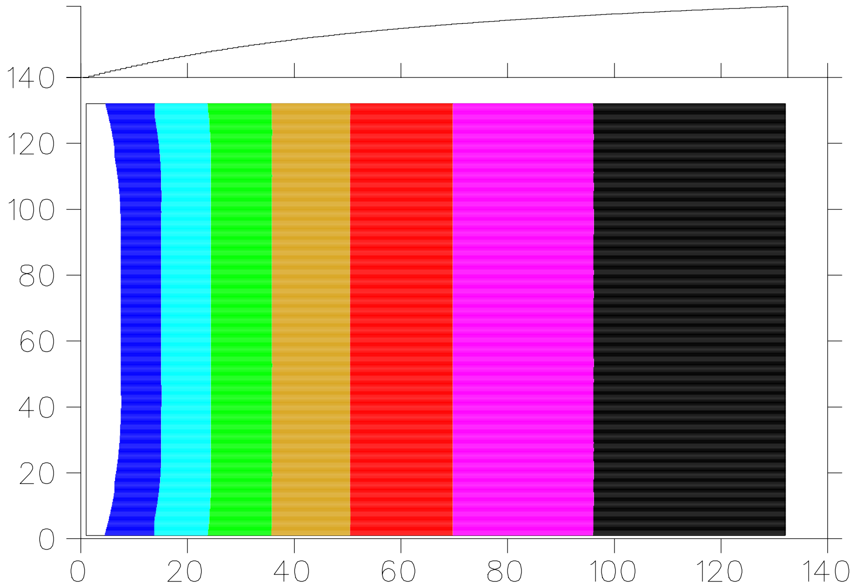

Figure 2: (Color)Plot of B_z vs $x = r\cos\theta$

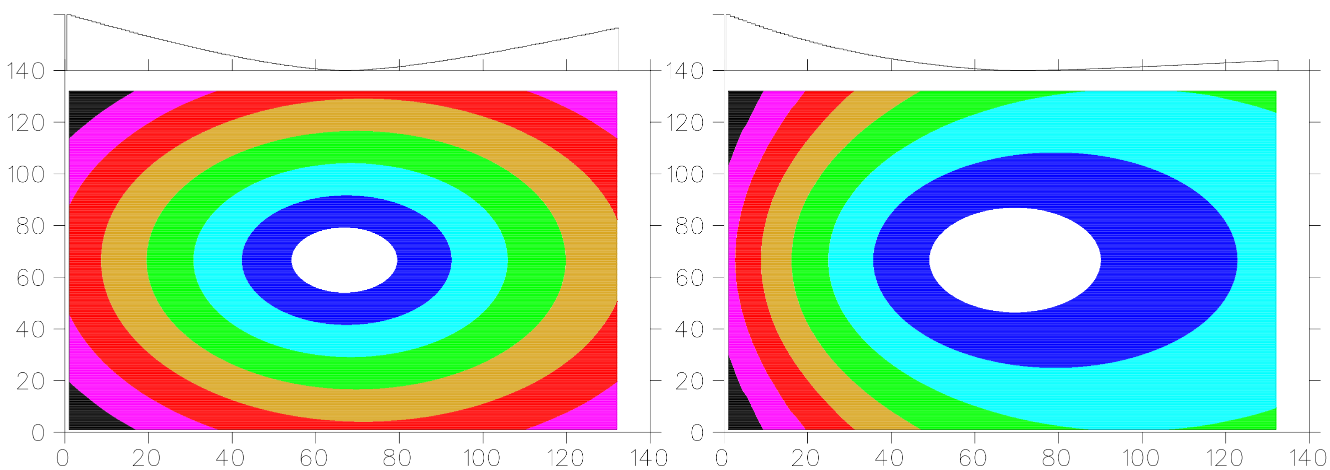

Figure 3: (Color)Polar plot of B_θ for $R_o = 1.20\ m$ (left) and $R_o = 0.20\ m$ (rigth); in both cases the radius of the lithium lens is $a = 0.10$ m.

[3] R. Fernow. ICOOL: a simulation code for ionization cooling of muon beams. In A. Luccio and W. MacKay, editor, *Proceedings of the 1999 Particle Accelerator Conference*, page 3020, 1999. Latest version is available at http://pubweb.bnl.gov/people/fernow/icool/readme.html.

[4] P. Morse and H. Feshbach Methods of Theoretical Physics, page 115 Mc Graw-Hill, New York, 1953