SOLENOIDAL COOLING LATTICES

Juan C. Gallardo, BNL, Upton, NY 11973, USA

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Abstract

The classification of periodic alternating lattices was analyzed in great details by R. Fernow [1]. In this note I suggest a faster and economical numerical procedure to calculate the matched beta (β) function of each one of these lattices.

1 INTRODUCTION

It is well known that a periodic lattice has resonances when the betatron wavelength Λ of a single particle is identical to the period of the periodic magnetic field λ; this happens at $p_s \approx \frac{q\lambda}{4\pi}\sqrt{< B^2 >} \approx \frac{\lambda B_o}{59}$. [2] In the vicinity of these values the transmission through the channel is reduced dramatically; consequently, there are regions in momentum space where the transmission is optimal also known as *pass bands.* It is of interest to calculate the beta function in the pass bands zones and in particular the matched β-function is of maximum concern.

The standard procedure is to solve the beta function equation of motion

$$\boxed{2\beta(s)\beta''(s) - \beta'(s)^2 + 4\beta(s)^2\kappa(s)^2 - 4 = 0} \tag{1}$$

where $\kappa(s) = \frac{qB_s(s)}{2p_s}$ and then imposing the constrains $\beta(0) \equiv \beta(\lambda)$ and $\beta'(0) \equiv \beta'(\lambda)$. In practice the above equation (Eq. 1) is repetitively solved for different initial conditions $\beta(0)$and$\beta'(\lambda)$ guided by an optimizer until the equality with the value at the end of the period is achieved (within an assumed error).

2 New Approach

In a recent paper H. Qin and R. Davidson [3] have pointed out that the generalized Courant-Snyder invariant (constant of motion) can be used to simplify the finding of matched solutions for periodic solenoidal lattices. The numerical algorithm is fast and simple and it requires to find the solution of Eq. 1 only once.

First I recall the main results of Ref. [3] without giving the mathematical proof

Theorem 2.1 *If $\omega_1(s)$ and $\omega_2(s)$ are the envelope functions ($\omega(s)^2 = \beta(s)\epsilon$ with ϵ the emittance of the beam) and $\kappa(s)$ is an arbitrary periodic function with period λ ,i.e $\kappa(s) = \kappa(s+\lambda)$ and the envelope function satisfy the equations*

$$\begin{aligned} \omega_1''(s) + \kappa(s)^2\omega_1(s) &= \frac{\epsilon_1^2}{\omega_1(s)^3} \\ \omega_2''(s) + \kappa(s)^2\omega_2(s) &= \frac{\epsilon_2^2}{\omega_2(s)^3} \end{aligned} \tag{2}$$

then

$$\boxed{I = \epsilon_2^2\left(\frac{\omega_1(s)}{\omega_2(s)}\right)^2 + \epsilon_1^2\left(\frac{\omega_2(s)}{\omega_1(s)}\right)^2 + \left(\omega_2(s)\omega_1'(s) - \omega_2'(s)\omega_1(s)\right)^2 \quad \textbf{\textit{is an invariant}}} \tag{3}$$

The presence of the invariant is, of course, the telling sign of a symmetry in the problem. References [3] discuss this in details and identify the symmetry group and the underlying Lee algebra. This symmetry is refers as the Courant-Snyder symmetry and the invariant Eq. 3 as the Courant-Snyder invariant. Several particular cases of this invariant has been discussed in the literature.

We give a few intermediate steps to verify that $\frac{dI}{ds} = 0$ (we take $\epsilon_1 = \epsilon_2 = 1$ to simplify the equations):

- $\frac{dI}{ds} = 2\left(\omega_2\omega_1' - \omega_2'\omega_1\right)\left\{\omega_2\omega_1'' - \omega_2''\omega_1\frac{\omega_2}{\omega_1^3} - \frac{\omega_1}{\omega_2^3}\right\}$ after judicious collection of terms
- substituting ω_1'' and ω_2'' from Eq. 2 we verify that indeed $\frac{dI}{ds} = 0$.

We define, as customary, the phase advance function

$$\Psi(s) = \int_o^s ds' \frac{\epsilon_2}{\omega_2(s')^2} \tag{4}$$

and we assert that

$$\boxed{\omega_1(s) = \frac{\omega_2(s)}{\sqrt{2\epsilon_2^2}} \sqrt{\left(I + \sqrt{I^2 - 4\epsilon_1^2\epsilon_2^2} \sin 2(\Psi(s) + \Phi_0)\right)}} \tag{5}$$

here I and Φ_0 are two constants. We understand Eq. 5 as a general solution of the envelope equation, *i.e* $\omega_1(s)$, in term of a particular solution $\omega_2(s)$ obtained with arbitrary initial conditions $\omega_2(0)$ and $\omega_2'(0)$.

Now we proceed to calculate the matched solution; to do that we look for constants I and Φ such that

$$\begin{aligned} &\omega_1(0) \equiv \omega_1(\lambda) \quad \text{and} \\ &\omega_1'(0) \equiv \omega_1'(\lambda) \end{aligned} \tag{6}$$

these two conditions yield two non-linear algebraic equations to be solved by standard root searching algorithms. Once I_{new} and Φ_{new} are determined then the matched envelope function and consequently the beta function is written as

$$\omega(s)^{\text{match}} = \frac{\omega_2(s)}{\sqrt{2\epsilon_2^2}} \sqrt{\left(I_{new} + \sqrt{I_{new}^2 - 4\epsilon_1^2\epsilon_2^2} \sin 2(\Psi(s) + \Phi_{new})\right)} \tag{7}$$

3 Results

A simple fortran program [4] has been written to integrate the envelope function equation and a subsequent one to solve the two non-linear algebraic equations.

The results are shown in Figs. 1, 2.

These results are in agreement with those found by Fernow [1], [2].

4 Acknowledgments

I would like to thank R. Fernow for his comments

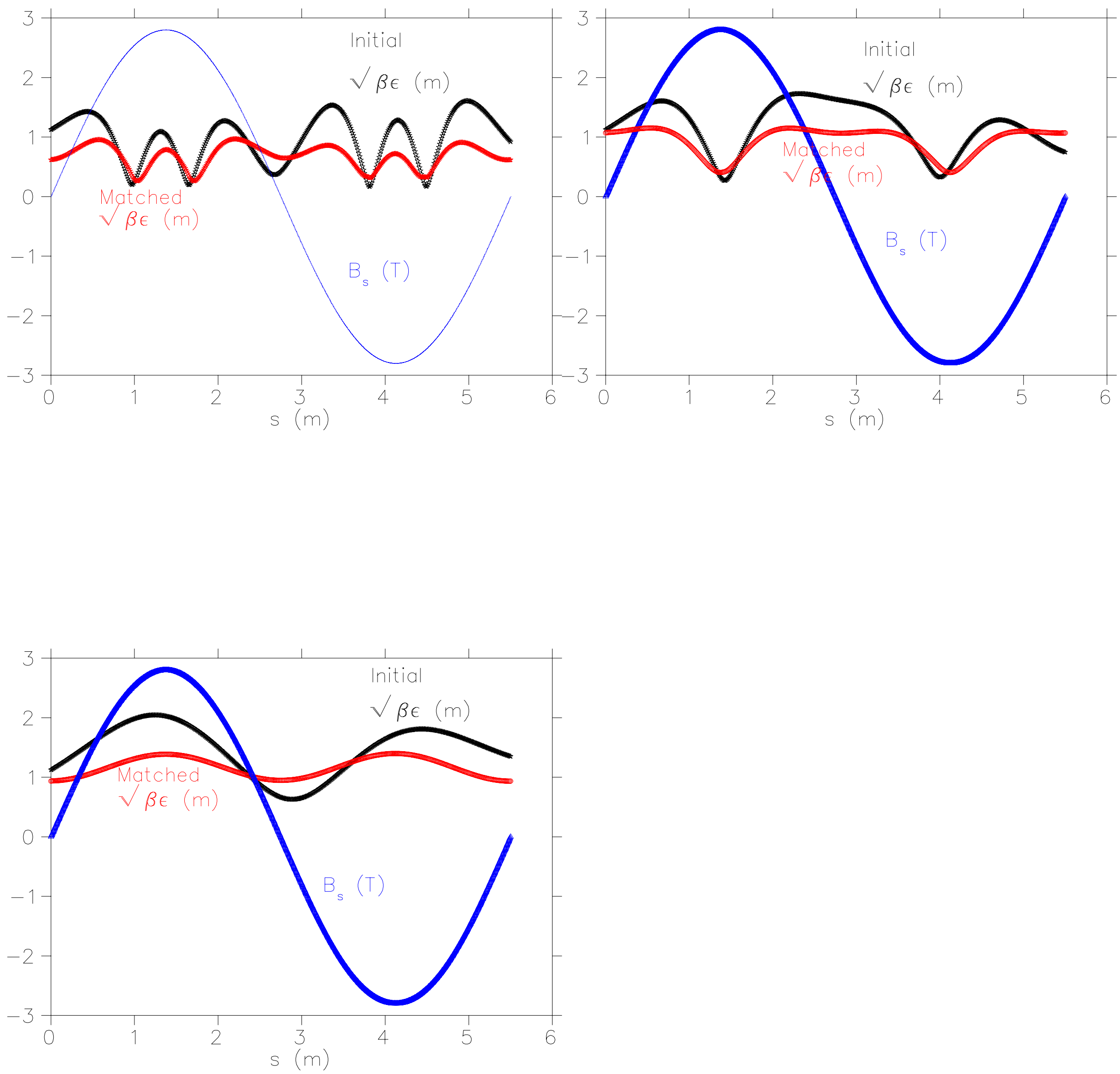

Figure 1: (Color)Plot of $\omega^{\mathrm{matched}}$ vs. s for different momentum from left to right, 90 , 170, and 400 MeV/c.

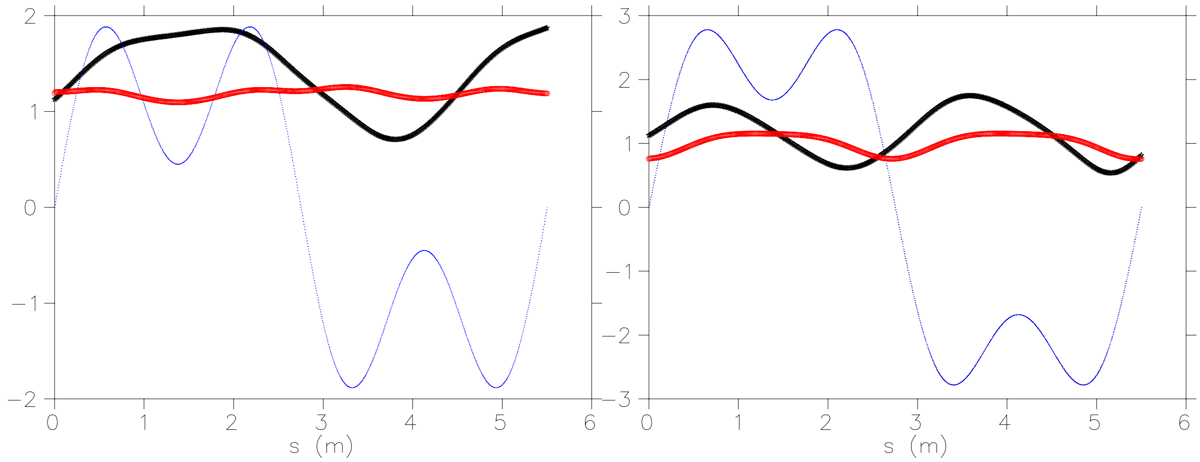

Figure 2: (Color)Plot of $\omega^{\mathrm{matched}}$ vs. s for different momentum from left to right, 275 and 380 MeV/c. Magnetic field with harmonics.

References

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[2] R. Fernow *Classification of Periodic Alternating-Solenoid Lattices* NFCMM-doc-244. http://nfmcc-docdb.fnal.gov:8080/cgi-bin/ShowDocument?docid=244

[3] H. Qin and R. Davidson *An Exact Magnetic-Moment Invariant of Charged-Particle Gyromotion* Phys.Rev.Lett., **96**,085003 (2006) *Symmetries and invariants of the oscillator and envelope equations with time-dependent frequency* Phys. Rev. STAB, **9** 054001 (2006)

[4] W. Press, S. Teukolsky, W. Vetterling and B. Flannery *Numerical recipes in FORTRAN: the art of scientific computing 2nd ed.* Cambridge University Press (1992). Both Fortran programs are available upon request