Tune Shift with Amplitude induced by Quadrupole Fringe Fields

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Abstract

Using Lie algebra techniques, we derive an analytical expression for the nonlinear Hamiltonian and the linear tune shift with amplitude due to quadrupole fringe fields. Numerical examples for the muon storage ring are compared with exact results from COSY INFINITY [1].

In current-free regions, the magnetic field fulfills $\vec{\nabla} \times \vec{B} = \vec{0}$ and $\vec{\nabla} \cdot \vec{B} = 0$. It can be derived either from a scalar potential ϕ or a vector potential $\vec{A}$, as $\vec{B} = \vec{\nabla} \times \vec{A} = \vec{\nabla}\phi$. If the field does not depend on z, the differential operators act only in the two transverse dimensions. In this case, the general form of the transverse magnet field is the standard multipole expansion:

$$B_y + iB_x = \sum_{n=1}^{\infty} [b_n + ia_n]\ [x + iy]^{n-1}/r_0^{n-1} \tag{1}$$

where r_0 denotes a reference radius. This is the usual situation without fringe fields. The longitudinal field component $B_z = B_{z0}$ is constant and equal to zero, except in a solenoid. The corresponding scalar potential for a normal quadrupole field ($b_2 \neq 0$) is

$$\Phi_2 = \frac{b_2}{r_0} xy, \tag{2}$$

for a normal octupole ($b_4 \neq 0$)

$$\Phi_{4n} = \frac{b_4}{r_0^3}[x^3 y - xy^3], \tag{3}$$

and for a skew octupole ($a_4 \neq 0$)

$$\Phi_{4s} = \frac{a_4}{4r_0^3}[y^4 - 6x^2y^2]. \tag{4}$$

Now consider a quadrupole of finite length and aperture, whose field depends on the longitudinal position z. In this case, the scalar potential Φ contains z-dependent terms and obeys the three-dimensional Laplace equation. In polar coordinates, $x = r\cos\theta$ and $y = r\sin\theta$, the scalar potential can be written as [2, 3]

$$\Phi(r,\theta,z) = G(r,z) r^2 \frac{\sin 2\theta}{2!} = [G_{20}(z) + G_{22}(z) r^2 + \ldots] r^2 \frac{\sin 2\theta}{2} \tag{5}$$

The first term in the square brackets on the right-hand side, $G_{20}(z)$, parametrises the field variation on the magnet axis, via $G_{20}(z) = \partial B_y/\partial x(z)|_{r=0}$. Its derivative also gives rise to a longitudinal field component.

The second term in Eq. (5) is related to the second derivative of G_{20}:

$$G_{22}(z) = -\frac{1}{12}\frac{d^2 G_{20}(z)}{dz^2} \tag{6}$$

The scalar potential associated with this term is proportional to

$$r^4 \frac{\sin 2\theta}{2} = [x^3 y + y^3 x] \tag{7}$$

Comparison with Eqs. (3) shows that this polynomial differs from that of an ordinary octupole by the relative sign of its two arguments. In addition, derivatives with respect to z introduce longitudinal field components, which are absent for fields that are independent of z. Thus, for a variety of reasons the fringe field effect cannot be described by the usual multipole expansion [4][1].

The integrated effect on a particle trajectory is conventionally described by a Hamiltonian which contains the vector potential $\vec{A}$ and not the scalar potential Φ. Thus, the polynomial form of the Hamiltonian form is different from that of the scalar potential.

For example, a normal quadrupole ($b_2 \neq 0$) corresponds to the Hamiltonian

$$H_{2n} = \frac{1}{2} K_{2n} [x^2 - y^2] \tag{8}$$

where $K_2 = b_2 l_Q/(B\rho)/r_0$, l_Q denotes the length of the magnet, and $(B\rho)$ the magnetic rigidity. Similarly, the Hamiltonians for a normal ($b_4 \neq 0$) or skew octupole ($a_4 \neq 0$) are

$$\begin{aligned} H_{4n} &= \frac{1}{24} K_{4n} [x^4 - 6x^2 y^2 + y^4], & (9) \\ H_{4s} &= \frac{1}{6} K_{4s} [y^3 x - x^3 y], & (10) \end{aligned}$$

with $K_{4n} = 6b_4/(B\rho)/r_0^3$, and $K_{4s} = 6a_4/(B\rho)/r_0^3$. The evolution of a particle trajectory then follows from Hamilton's equations: $dx'/dz = -\partial H/\partial x$, and $dy'/dz = -\partial H/\partial y$.

To represent the fringe field effect by a Hamiltonian, we must find the vector potential $\vec{A}$. For simplicity, we rewrite Eq. (5) as

$$\Phi(r, \theta, z) = \Phi_0 \sin 2\theta \tag{11}$$

so that only the quadrupolar azimuthal dependence is explicit. We know that $B_r = \partial\Phi/\partial r$, $B_z = \partial\Phi/\partial z$, and $B_\theta = 1/r(\partial\Phi/\partial\theta)$. One choice of vector potential which gives the same magnetic field is [5]

$$\begin{aligned} A_r &= \frac{1}{2} r \frac{\partial\Phi_0}{\partial s} \cos 2\theta & (12) \\ A_z &= -\frac{1}{2} r \frac{\partial\Phi_0}{\partial r} \cos 2\theta & (13) \\ A_\theta &= 0 & (14) \end{aligned}$$

This can be verified explicitly:

$$\begin{aligned} B_r &= \frac{1}{r}\frac{\partial A_z}{\partial\theta} = \frac{\partial\Phi_0}{\partial r} \sin 2\theta = \frac{\partial\Phi}{\partial r} & (15) \\ B_z &= -\frac{1}{r}\frac{\partial A_r}{\partial\theta} = \frac{\partial\Phi_0}{\partial z} \sin 2\theta = \frac{\partial\Phi}{\partial z} & (16) \\ B_\theta &= \frac{\partial A_r}{\partial z} - \frac{\partial A_s}{\partial r} = \frac{1}{r}\frac{\partial}{\partial\theta} \Phi_0 \sin 2\theta & (17) \end{aligned}$$

[1]By placing several families of octupoles at positions with large and small β_x/β_y ratios, respectively, it might still be possible using octupoles to globally compensate the two terms proportional to $x^3 y$ and $y^3 x$.

where in the last line we used the fact that the scalar potential $\Phi = \Phi_0 \sin 2\theta$ satisfies the Laplace equation:

$$\frac{\partial^2 \Phi}{\partial \theta^2} + \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\right)\Phi + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}\Phi = 0 \tag{18}$$

In the following, we only retain the lowest-order terms generated by G_{20} (in A_r) and G_{22} (in A_z), namely:

$$A_r \approx \frac{r^3}{4}\left[\frac{dG_{20}(z)}{dz} + r^2\frac{dG_{22}(z)}{dz} + \ldots\right]\cos 2\theta \tag{19}$$

$$A_z \approx -\frac{r^2}{2}\left[G_{20}(z) + 2r^2 G_{22}(z) + \ldots\right]\ \cos 2\theta \tag{20}$$

which are inserted into the general form of the Lie-algebraic Hamiltonian [5]:

$$H = \frac{1}{2}\left(p_r - \frac{qA_r}{p_0(1+\delta)}\right)^2 + \frac{p_\theta^2}{2r^2} - \frac{qA_z}{p_0(1+\delta)} \tag{21}$$

$$\approx H_{\rm lin} - \frac{q}{p_0}p_r A_r - \frac{q}{p_0}A_z \tag{22}$$

Here q denotes the charge of the particle, and p_r, p_θ the radial and angular momenta, respectively, and we omit the δ-dependence. Keeping again only the two lowest-order nonlinear terms (up to 4th power in r and p_r) we obtain

$$H \approx H_{\rm lin} - \frac{1}{B\rho}\frac{dG_{20}}{dz}\frac{1}{4}r^3 p_r \cos 2\theta + r^4\frac{1}{B\rho}G_{22}(z)\cos 2\theta \tag{23}$$

where the linear part, $H_{\rm lin}$, includes the usual kinematic term, $\frac{1}{2}[p_r^2 + p_\theta^2/r^2]$, and also the linear quadrupole focusing, $\frac{1}{2}K_Q r^2 \cos 2\theta$, with $K_Q = G_{20}/(B\rho)$. The nonlinear perturbation, $H_{\rm pert} = H - H_{\rm lin}$, can be expressed in cartesian coordinates, x and y as

$$H_{\rm pert} \approx -\frac{1}{B\rho}\frac{dG_{20}(z)}{dz}\frac{1}{4}(x^2 - y^2)(xp_x + yp_y) - \frac{1}{12}\frac{1}{B\rho}\frac{d^2G_{20}}{dz^2}(x^4 - y^4) \tag{24}$$

Next we integrate the Hamiltonian over the incoming or outgoing side of the magnet. We assume that the fringe field extends over a longitudinal distance $\pm\Delta$ around the edge of the magnet. The distance Δ is proportional to the magnet aperture. Next, to evaluate the integral

$$\hat{H} = \int_{-\Delta}^{\Delta} H_{\rm pert}(z)dz \tag{25}$$

we perform a Taylor expansion of the transverse coordinates in terms of z, around the entrance or exit points of the magnet [2]. The two reference points are taken to be the positions where the field gradient is 1/2 of its value at the center of the magnet. We assume that the field fall-off is symmetric about each of these points.

For example, the second argument in Eq. (24) is expanded as

$$(x^4 - y^4) = \left\{[x^4 - y^4]_0 + z\left[\frac{d}{dz}(x^4 - y^4)\right]_0 + \frac{z^2}{2}\left[\frac{d^2}{dz^2}(x^4 - y^4)\right]_0 + \ldots\right\} \tag{26}$$

[2]For a special form of the field fall-off, and considering one dimension only, M. Venturini recently computed the integrated fringe-field Hamiltonian without resorting to a Taylor map expansion [6].

The subindex 0 refers to the expansion point. Inserting this into Eq. (25), we obtain integrals from $-\Delta$ to $+\Delta$ of the form

$$\begin{aligned}
\frac{1}{B\rho}\int_{-\Delta}^{\Delta} G'_{20} dz &= K_Q, &(27)\\
\frac{1}{B\rho}\int_{-\Delta}^{\Delta} G'_{20} z dz &= 0, &(28)\\
\frac{1}{B\rho}\int_{-\Delta}^{\Delta} G'_{20} z^2 dz &\approx \frac{1}{3}\Delta^2 K_Q, &(29)\\
\frac{1}{B\rho}\int_{-\Delta}^{\Delta} G''_{20} dz &= 0, &(30)\\
\frac{1}{B\rho}\int_{-\Delta}^{\Delta} G''_{20} z dz &= -K_Q, &(31)\\
\frac{1}{B\rho}\int_{-\Delta}^{\Delta} G''_{20} z^2 dz &= 0, &(32)
\end{aligned}$$

where we used the assumption that the fringe fall-off is symmetric about the entrance (or exit) point. All the results quoted are for the incoming edge. For the outgoing edge, the signs on the right-hand-side are inverted.

Three terms, corresponding to the three non-vanishing integrals above, contribute to the integral, Eq. (25). We make this transparent by writing $\hat{H} = \hat{H}_1 + \hat{H}_2 + \hat{H}_3$. The first term results from the first term on the right-hand-side of Eq. (24) and the nonzero integral in Eq. (27). It reads

$$\hat{H}_1 = -\frac{1}{4}K_Q\left[(x^2-y^2)_i(xp_x+yp_y)_i - (x^2-y^2)_o(xp_x+yp_y)_o\right] \tag{33}$$

The subindices i and o indicate coordinates at the incoming and outgoing sides, respectively, and K_Q is the normalized quadrupole gradient in units of inverse squarea meters, or $K_Q = G_{20}(0)/(B\rho)$.

The second term arises from the second term in Eq. (240 and from Eq. (31):

$$\hat{H}_2 = \frac{1}{3}K_Q\left[(x^3p_x - y^3p_y)_i - (x^3p_x - y^3p_y)_o\right] \tag{34}$$

Adding the two previous equations, we get

$$\begin{aligned}
\hat{H}_{1+2} &= \frac{1}{12}K_Q\left[xp_x(x^2+3y^2) - yp_y(y^2+3x^2)\right]_i\\
&\quad -\frac{1}{12}K_Q\left[xp_x(x^2+3y^2) - yp_y(y^2+3x^2)\right]_o
\end{aligned} \tag{35}$$

This agrees with the effect of an ideal hard-edge fringe field, which was calculated by Lee-Whiting [7] and, more recently and in more general form by E. Forest and J. Milutinovic [8]. This term, which is independent of the fringe field length Δ, will turn out to be the dominant nonlinear effect, in good agreement with Venturini's result for a 1-dimensional fringe field [6].

Finally, the last term, which derives from the integral in Eq. (29) and, again, from the first part of Eq. (24), depends on the fringe length:

$$\hat{H}_3 = -\frac{1}{24}\Delta^2 K_Q\left[\left(6xp_x^3 - 6yp_y^3 - 10K_Qx^3p_x - 10K_Qy^3p_y\right.\right.$$

$$
\begin{aligned}
&+6K_Qx^2yp_y+6K_Qy^2xp_x+2xp_xp_y^2-2yp_yp_x^2\big)_i \\
&-\big(6xp_x^3-6yp_y^3-10K_Qx^3p_x-10K_Qy^3p_y \\
&+6K_Qx^2yp_y+6K_Qy^2xp_x+2xp_xp_y^2-2yp_yp_x^2\big)_o\Big]
\end{aligned} \tag{36}
$$

The coordinates at the outgoing side, 'o', can be expressed by those at the entrance of the magnet using the linear transformation through the quadrupole. We assume that $(\sqrt{K_Q}l_Q)$ is sufficiently small, that we can linearly expand the $\sin(\sqrt{K_Q}l_Q)$ or $\sinh(\sqrt{K_Q}l_Q)$ functions in the elements of the R matrix. In our example below, $\sqrt{K_Q}l_Q$ is about 0.13. We will also assume that the quadrupole is short, and that the beta function at the quadrupole is large, or specifically that

$$K_Q \gg 1/\beta^2 \tag{37}$$

and

$$l_Q \ll \beta \tag{38}$$

Under these conditions, the transverse coordinates are approximately constant within the magnet

$$x_0 \approx x_i \tag{39}$$

$$y_0 \approx y_i \tag{40}$$

and primarily only the trajectory slopes change, roughly as

$$p_{xo} \approx p_{xi}-(K_Ql_Q)x_i \tag{41}$$

$$p_{yo} \approx p_{yi}+(K_Ql_Q)y_i \tag{42}$$

The Δ-independent part of $\hat{H}$ becomes

$$\hat{H}_{1+2} \approx \frac{1}{12}(K_Ql_Q)K_Q\left[x^4+6x^2y^2+y^4\right] \tag{43}$$

where the coordinates x and y may now be taken to be those at the center of the magnet.

Again using Eqs. (41) and (42) and keeping only the largest components, the next term in the Hamiltonian, Eq. (36), can be approximated as

$$\hat{H}_3 \approx \frac{5}{12}\Delta^2K_Q^2(K_Ql_Q)\left[x^4-y^4\right] \tag{44}$$

Expressing the transverse positions in terms of action angle coordinates, $x=\sqrt{2I_x\beta_x}\cos\phi_x$ and $y=\sqrt{2I_y\beta_y}\cos\phi_y$, and averaging the Hamiltonian $\hat{H}=(\hat{H}_{1+2}+\hat{H}_3)$ over the betatron phases ϕ_x and ϕ_y using $<\cos^4\phi>=3/8$ and $<\cos^2\phi>=1/2$, the nonlinear Hamiltonian representing the effect of the fringe fields reads

$$
\begin{aligned}
<\hat{H}> \; = \; & \frac{1}{8}\sum_Q(K_Ql_Q)K_Q\left[\beta_{x,Q}^2I_x^2+4\beta_{x,Q}\beta_{y,Q}I_xI_y+\beta_{y,Q}^2I_y^2\right] \\
&+\frac{5}{8}\sum_Q\Delta^2K_Q^2(K_Ql_Q)[\beta_x^2I_x^2-\beta_y^2I_y^2]
\end{aligned} \tag{45}
$$

The sum is over all quadrupoles Q, and $K_Q > 0$ for a horizontally focusing quadrupole. The derivatives of $< \hat{H} >$ with respect to $I_{x,y}$ yield the amplitude dependent tune shifts:

$$\begin{aligned}\Delta Q_x &= \frac{1}{8\pi}\sum_Q (K_Q l_Q) K_Q \left[\beta_{x,Q}^2 I_x + 2\beta_{x,Q}\beta_{y,Q} I_y\right] \\ &\quad + \frac{5}{8\pi}\sum_Q \Delta^2 l_Q K_Q^3 \beta_x^2 I_x \qquad (46)\end{aligned}$$

$$\begin{aligned}\Delta Q_y &= \frac{1}{8\pi}\sum_Q (K_Q l_Q) K_Q \left[\beta_{y,Q}^2 I_y + 2\beta_{x,Q}\beta_{y,Q} I_x\right] \qquad (47) \\ &\quad - \frac{5}{8\pi}\sum_Q \Delta^2 l_Q K_Q^3 \beta_y^2 I_y . \qquad (48)\end{aligned}$$

where N_Q is the number of quadrupoles. All three tune shifts, $\Delta Q_x/\Delta I_x$, $\Delta Q_x/\Delta I_y = \Delta Q_y/\Delta I_x$, and $\Delta Q_y/\Delta I_y$, are positive and of comparable magnitude.

As a practical example, consider the muon storage ring, whose optics is shown in Fig. 1. The ring consists of three parts: a neutrino production straight, a return straight, and the (two) arcs. We first evaluate the tune shift from the arcs. A detailed view of the arc optics is shown in Fig. 2. There are a total of 31 arc cells, each comprising two quadrupoles. Using maximum and minimum beta functions of $\beta_{x,y}$ of 16 m and 3 m, respectively, a quadrupole length $l_Q = 1$ m, strength $K_Q = 0.31$ m^{-2}, and zero fringe extent ($\Delta = 0$), we estimate $\Delta Q_x/\Delta I_x \approx \Delta Q_y/\Delta I_y \approx 31$, and $\Delta Q_x/\Delta I_y = \Delta Q_y/\Delta I_x \approx 23$. We can compare these estimates with an exact calculation using the program COSY INFINITY [1, 9], which gives $\Delta Q_x/\Delta I_x = 30$, $\Delta Q_x/\Delta I_y = 28$ and $\Delta Q_y/\Delta I_y = 34$. The agreement between COSY and our first-order estimate is quite satisfactory.

The same comparison can be made for the neutrino production straight. Here the maximum and minimum beta functions are about 430 m and 300 m, the quadrupole strength $K_Q \approx 0.0019$ m^{-2}, the length $l_Q = 3$ m, and the total number of cells is 5. We then obtain $\Delta Q_x/\Delta I_x \approx \Delta Q_y/\Delta I_y \approx 0.6$, and $\Delta Q_x/\Delta I_y = \Delta Q_y/\Delta I_x \approx 1.1$. These values almost perfectly agree with the COSY results of 0.6 and 1.0, respectively. The product $[\beta^2 K_Q^2 l_Q]$ scales about as $1/\beta$, which explains why the tune shift induced in the arcs is much larger than that from the production straight.

The actual value of the tune shift at 1σ can be estimated by setting $I_{x,y}$ in the above expressions for $\Delta Q_{x,y}$ equal to half the rms geometric emittance $\epsilon_{x,y}/2$. For the nominal rms emittance, $\epsilon_{x,y} \approx 7$ μm, the tune shift due to fringe fields in arcs and straight section is small.

The part of the tune shift quadratic in Δ is suppressed compared to the Δ-independent part by a factor $\Delta^2 K_Q$. For quadrupoles in the production straight with $K_Q = 0.002$ m^{-2} and $\Delta = 0.17$ m, this suppression factor is 10^{-4}. We expect that the contributions from higher-order terms are even less important.

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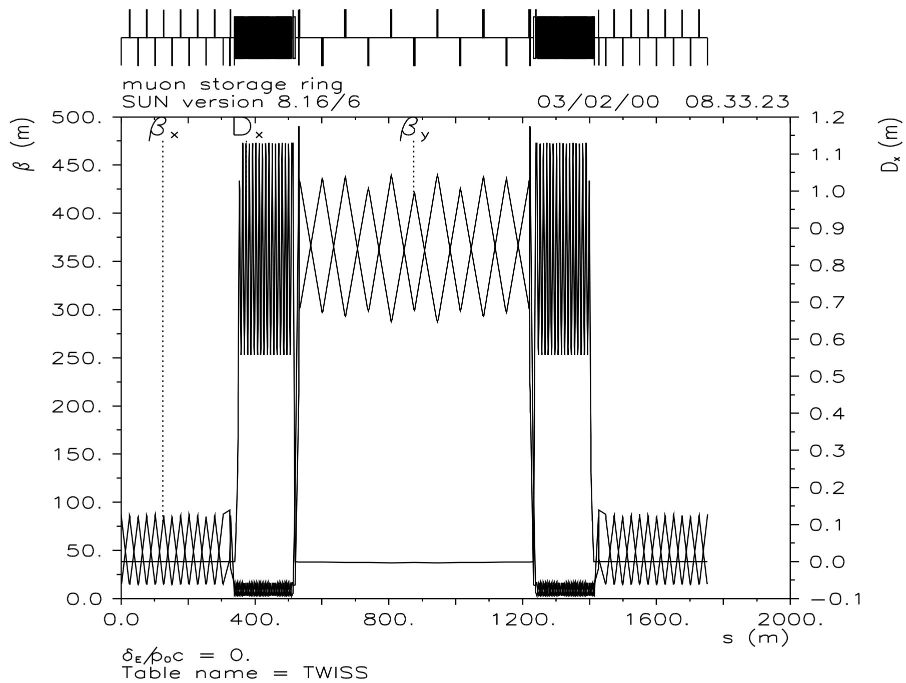

Figure 1: Optics for the FNAL muon storage ring; courtesy of C. Johnstone.

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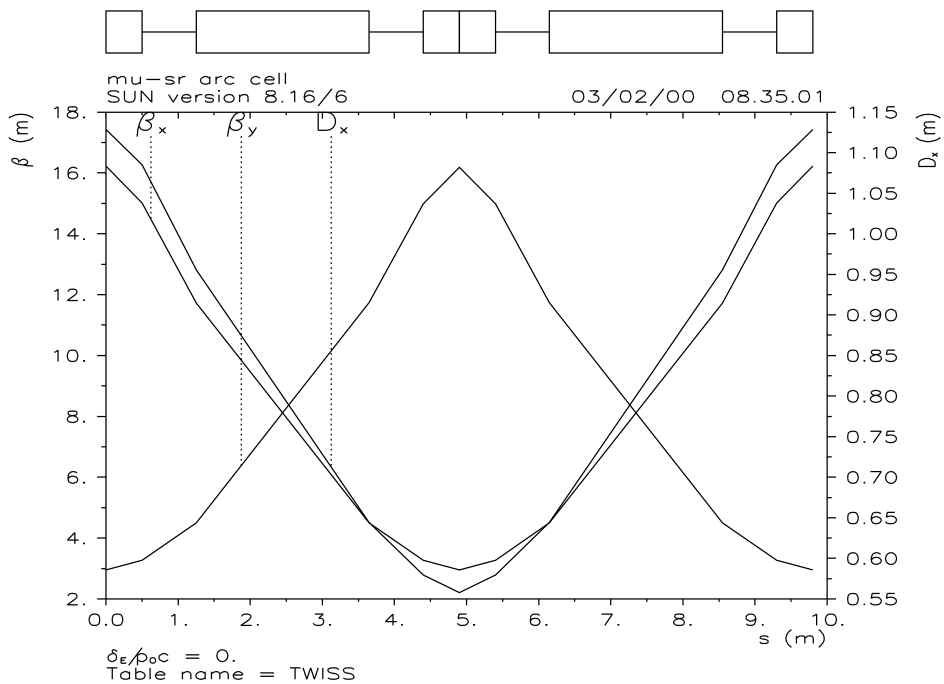

Figure 2: Optics for an arc cell of the FNAL muon storage ring; courtesy of C. Johnstone.